12-12:50pm **Midterm Exam 2**

Fall 2012 Math 0220

100 points total **S o l u t i o n s**

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1. Find a formula for the inverse of the function $f(x) = \dfrac{2x+3}{4x-1}$.

Solution: $y = \dfrac{2x+3}{4x-1}$, $x = \dfrac{2y+3}{4y-1}$.

Solve for y: $x(4y-1) = (2y+3)$, $4xy - x = 2y+3$, $4xy - 2y = x+3$, $y(4x-2) = x+3$, $y = \dfrac{x+3}{4x-2}$.

Hence $f^{-1}(x) = \dfrac{x+3}{4x-2}$

2. Use logarithmic differentiation to find y' if $y = x^{\sin x}$.

Solution: $\ln y = \ln(x^{\sin x})$, $\ln y = \sin x \ln x$, $\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(\sin x \ln x)$,

$\dfrac{y'}{y} = \cos x \ln x + \sin x \cdot \dfrac{1}{x}$, $y' = y\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$,

$$y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

3. The half-life of cesium-137 is 30 years. If you have a 200-mg sample after how long will only 30 mg remains? Simplify your answer.

Solution: Let $m(t)$ be the mass of cesium-137 that remains after t years. Then

$m(t) = m(0)\,e^{kt}$, $m(30) = m(0)\,e^{30k} = \dfrac{1}{2}\,m(0)$.

The last equations gives $e^{30k} = \dfrac{1}{2}$ or $e^k = \left(\dfrac{1}{2}\right)^{1/30} = 2^{-1/30}$.

Hence, $e^{kt} = 2^{-t/30}$ and $m(t) = m(0)\,2^{-t/30} = 200 \cdot 2^{-t/30}$.

To find after how long only 30 mg will remain we solve the equation

$200 \cdot 2^{-t/30} = 30 \Leftrightarrow 2^{-t/30} = 3/20 \Leftrightarrow -t/30 = \log_2(3/20)$

$t = -30\log_2(3/20)$ or $t = 30\log_2(20/3) = 30 \cdot \dfrac{\ln(20/3)}{\ln 2} = 30 \cdot \dfrac{\ln(3/20)}{\ln(1/2)}$ years.

4. Find the limit. Use l'Hospital rule if appropriate.

$\lim_{x\to 0^+} (2x)^x$.

Solution: It is an indeterminate form of type "$\frac{0}{0}$". $(2x)^x = e^{\ln(2x)^x} = e^{x\ln(2x)}$

$$\lim_{x\to 0^+} (2x)^x = \lim_{x\to 0^+} e^{x\ln(2x)} = e^{\lim_{x\to 0^+} x\ln(2x)}$$

$$\lim_{x\to 0^+} x\ln(2x)\ \left[\text{"}0\cdot\infty\text{"}\right] = \lim_{x\to 0^+} \frac{\ln(2x)}{x^{-1}}\ \left[\text{"}\frac{\infty}{\infty}\text{"}\right] \overset{H}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} \frac{x}{-1} \overset{DSP}{=} 0.$$

Hence, $\lim_{x\to 0^+} (2x)^x = e^0 = 1$.

5. Find the absolute maximum and the absolute minimum values of the function $f(x) = \dfrac{x}{x^2+1}$ on the interval $[0, 3]$.

Solution: The function $f(x)$ is continuous on $[0, 3]$ and we can apply the closed interval method.

CNs: $f'(x) = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

$f'(x) = 0 \Leftrightarrow 1 - x^2 = 0 \Leftrightarrow x = 1$ (the other root $x = -1$ is not in the interval).

$f'(x)$ exists everywhere. The only CN is $x = 1$ and $f(1) = \dfrac{1}{2}$.

$f(0) = 0,\ f(3) = \dfrac{3}{10}$.

The absolute maximum value is $\dfrac{1}{2}$, the absolute minimum value is 0.

6. Find two positive numbers whose product is 4 and whose sum of their squares is as small as possible. For a proof use the optimization method.

Solution: Let x and y be the numbers. Then $xy = 4$ and the sum of squares is $f(x) = x^2 + y^2$.

From the equation for the product we obtain $y = \dfrac{4}{x} = 4x^{-1}$. Then $f(x) = x^2 + 16x^{-2}$.

The problem is to minimize the function $f(x) = x^2 + 16x^{-2}$, when $x > 0$.

CNs: $f'(x) = 2x - 32x^{-3} = 2x - \dfrac{32}{x^3} = \dfrac{2(x^4-16)}{x^3}$. $f'(x) = 0$ whenever $x^4 = 16$ or $x^2 = 4$. Then $x = 2$ since $x > 0$.

$f'(x)$ DNE when $x = 0$ which is not in the domain of the function. Hence $f(x)$ has the only CN at 2.

The second derivative test: $f''(x) = 2 + \frac{3 \cdot 32}{x^4} > 0$ for all $x > 0$. Specifically $f''(2) > 0$. Hence there is a local minimum at 2. The function is concave up on its domain and has the only local minimum. Therefore this local minimum at $x = 2$ must be the absolute minimum.

Hence $x = 2$ and $y = \frac{4}{2} = 2$.

The numbers are 2 and 2.

7. Use Newton's method to approximate the number $\sqrt[3]{10}$. Use $x_1 = 2$ as an initial approximation and find the second approximation x_2 to the number.

Solution: Let $x = \sqrt[3]{10}$, then $x^3 = 10 \Leftrightarrow x^3 - 10 = 0$. Denote $f(x) = x^3 - 10$. Hence the number the number $\sqrt[3]{10}$ is the root of the equation $f(x) = 0$ and we apply Newton's method to find its approximation.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, with $x_1 = 2$.

$$f'(x) = 3x^2 \quad \Rightarrow \quad x_{n+1} = x_n - \frac{x_n^3 - 10}{3x_n^2} = \frac{2x_n^3 + 10}{3x_n^2} = \frac{2}{3}x_n + \frac{10}{3x_n^2}$$

$$x_2 = \frac{2}{3}x_1 + \frac{10}{3x_1^2} = \frac{2}{3} \cdot 2 + \frac{10}{3 \cdot 2^2} = \frac{4}{3} + \frac{10}{12} = \frac{13}{6} = 2\tfrac{1}{6}$$

bonus problem. Show that the equation $2\cos x - 5x = 1$ has exactly one real root.

Solution: The equation $2\cos x - 5x = 1$ is equivalent to the equation $2\cos x - 5x - 1 = 0$. Consider the function $f(x) = 2\cos x - 5x - 1$. $f(0) = 2 - 1 = 1 > 0$, $f(\pi/2) = 0 - 5\pi/2 - 1 < 0$. On the interval $[0, \pi/2]$ the function is continuous and satisties the inequality $f(0) > 0 > f(\pi/2)$. Applying the IVT with $N = 0$ we get that there is c in $(0, \pi/2)$ such that $f(c) = 0$, which means that c is a real root of the given equation.

Assume that the equation and hence the function has two or more roots. Let a and b be two roots. Then $f(a) = f(b) = 0$. The function is continuous and differentiable everywhere on $(-\infty, \infty)$. By the MVT there is c such that $f'(c) = 0$. But $f'(x) = -2\sin x - 5 \leq -3 < 0$ for all x. It contradicts to the fact that at c the derivative is zero. Hence the assumption about the existence of at least two roots was wrong. Therefore, the function and hence the equation has exactly one root.