10-10:50am

Midterm Exam 1

Spring 2012 Math 0220

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

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1. (a) (5 points) Determine whether $f(x) = x \tan x$ is even, odd or neither.

   (b) (5 points) Find the domain of the function $f(x)$.

(a) $f(-x) = (-x)\tan(-x) = -x(-\tan x) = x \tan x = f(x)$

$\Rightarrow f(x)$ is even.

(b) From $\tan x$ we get $x \neq \frac{\pi}{2} + \pi n$.

n is integer.

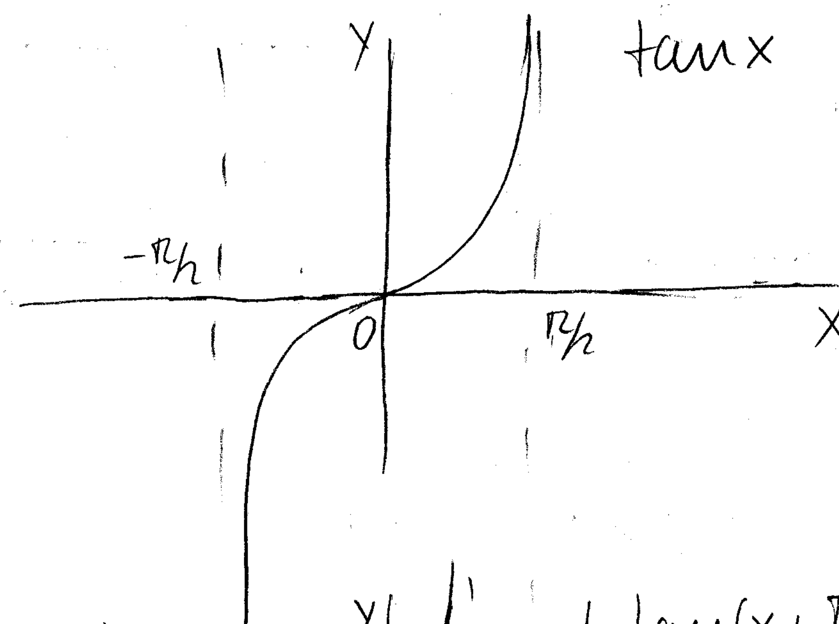
Hence $D = \{x \in \mathbb{R} \text{ except } x = \frac{\pi}{2} + \pi n\}$

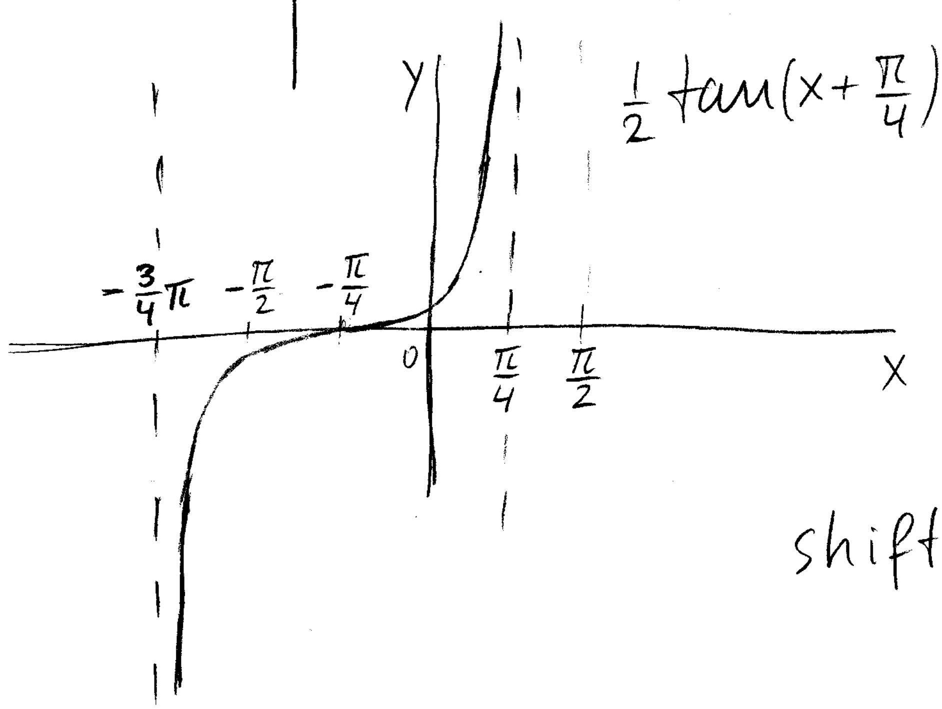
or all reals except $\frac{\pi}{2} + \pi n$

2. (10 points) Graph the function

$$f(x) = \frac{1}{2}\tan\left(x + \frac{\pi}{4}\right)$$

by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. Mark all essential points on the axes.





shift by $\frac{\pi}{4}$ to the left

compression by 2 along the y-axis

3. (15 points) Evaluate the limit, if it exists. If it does not exist explain why.

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{3x-15}$$

In your work mention what Rules, Laws, Theorems or Formulas you use.

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{3x-15} = \lim_{x\to 5} \frac{\sqrt{x-1}-2}{3(x-5)} \cdot \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$$

$$= \lim_{x\to 5} \frac{x-1-2^2}{3(x-5)(\sqrt{x-1}+2)} = \lim_{x\to 5} \frac{1}{3(\sqrt{x-1}+2)}$$

$$\underset{\text{DSP}}{=} \frac{1}{3(\sqrt{5-1}+2)} = \frac{1}{3(2+2)} = \boxed{\frac{1}{12}}$$

To show that a function has removable discontinuity at $x=a$ we have to show that it is discontinuous at a and $\lim_{x\to a} f(x)$ exists.

4. (a) (10 points) Does the function $f(x) = \dfrac{x^2+x-6}{3x+9}$ have removable discontinuity at -3. Support your answer.

$f(-3)$ DNE because -3 is not in the domain of $f(x)$ $\Rightarrow$ $f(x)$ is discontinuous at -3.

On the other hand,

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{(x+3)(x-2)}{3(x+3)} = \lim_{x\to -3} \frac{x-2}{3}$$

$$\overset{DSP}{=} \frac{-3-2}{3} = -\frac{5}{3}$$ exists and hence the discontinuity is removable.

(b) (5 points) If the discontinuity is removable, find a function $g(x)$ that agrees with $f(x)$ for $x \neq -3$ and is continuous at -3.

either

$$g(x) = \begin{cases} \dfrac{x^2+x-6}{3x+9}, & x \neq -3 \\ -\dfrac{5}{3}, & x = -3 \end{cases}$$

or

$$g(x) = \frac{x-2}{3}$$

5. (10 points) Using the definition of the derivative find $f'(x)$ if

$$f(x) = \sqrt{1 + 2x}$$

No credit be given if the derivative will be found without using the definition.

Sln $f'(x) = \lim_{h\to 0} \dfrac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h}$

$= \lim_{h\to 0} \dfrac{\sqrt{1+2x+2h} - \sqrt{1+2x}}{h} \cdot \dfrac{\sqrt{1+2x+2h} + \sqrt{1+2x}}{\sqrt{1+2x+2h} + \sqrt{1+2x}}$

$= \lim_{h\to 0} \dfrac{1+2x+2h-1-2x}{h(\sqrt{1+2x+2h} + \sqrt{1+2x})} = \lim_{h\to 0} \dfrac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}}$

$\underset{DSP}{=} \dfrac{2}{\sqrt{1+2x} + \sqrt{1+2x}} = \boxed{\dfrac{1}{\sqrt{1+2x}}}$

6. (15 points) Find an equation of the normal line to the curve $y = x\sqrt{x}$ at the point which x-coordinate is 4.

Sln $x = 4 \Rightarrow y = 4\sqrt{4} = 8$

slope $m = y'$ at 4 (tan. line)

$y' = (x^{3/2})' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$y'(4) = \frac{3}{2} \cdot \sqrt{4} = 3 = m$

Slope of the normal line is.

$-\frac{1}{m} = -\frac{1}{3}$

Normal line: $y = 8 - \frac{1}{3}(x-4)$

$$\boxed{y = -\frac{1}{3}x + 9\frac{1}{3}}$$

7. (10 points) For the function $f(x) = \frac{3x^2}{x+1}$ find $f''(1)$.

Sln: $f(x) = 3x^2(x+1)^{-1}$

$$f'(x) \underset{\substack{\text{Product} \\ \text{Power,} \\ \text{chain Rules}}}{=} 6x(x+1)^{-1} + 3x^2(-1)(x+1)^{-2}$$

$$f'(x) = 6x(x+1)^{-1} - 3x^2(x+1)^{-2}$$

$$f''(x) = (f'(x))' = 6(x+1)^{-1} - 6x(x+1)^{-2} - 6x(x+1)^{-2} + 6x^2(x+1)^{-3}$$

$$f''(1) = \frac{6}{2} - \frac{6 \cdot 1}{2^2} - \frac{6 \cdot 1}{2^2} + \frac{6 \cdot 1^2}{2^3} = 3 - \frac{3}{2} - \frac{3}{2} + \frac{3}{4}$$

$$\boxed{f''(1) = \frac{3}{4}}$$

Quotient Rule gives $f''(x) = \frac{6(x+1)^2 - 12x(x+1) + 6x^2}{(x+1)^3}$

or $f''(x) = \frac{6}{(x+1)^3}$

8. (15 points) Find $\frac{dy}{dx}$ by implicit differentiation if $7 + x = \cos(xy^2)$.

We dif-te both sides w.r.t. x and apply Chain and Product Rules to get:

$$\frac{d}{dx}[7+x] = \frac{d}{dx}[\cos(xy^2)]$$

$$1 = -\sin(xy^2)\cdot(y^2 + x\cdot 2yy')$$

$$1 = -y^2\sin(xy^2) - 2xy\sin(xy^2)\cdot y'$$

$$2xy\sin(xy^2)\cdot y' = -1 - y^2\sin(xy^2)$$

$$\boxed{y' = -\frac{1+y^2\sin(xy^2)}{2xy\sin(xy^2)}}$$

or $$y' = -\frac{\frac{1}{\sin(xy^2)} + y^2}{2xy}$$

bonus problem [8 points extra] Two curves are orthogonal if their tangent lines are perpendicular at each point of intersection.
Show that the curves $x^2 + y^2 = 5$ and $x + 2y = 0$ are orthogonal.

Points of intersection of the curves:

$x = -2y$, $(-2y)^2 + y^2 = 5$, $5y^2 = 5$, $y = \pm 1$

$x = \mp 2$, points: $(-2, 1)$, $(2, -1)$

1st curve: $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y} = m_1$

2nd curve: $1 + 2y' = 0 \Rightarrow y' = -\frac{1}{2} = m_2$

Two tan lines are perpendicular means that $m_1 = -\frac{1}{m_2} = 2$ or $-\frac{x}{y} = 2$. This equality holds for both points of intersection: $-\frac{-2}{1} = 2$ and $-\frac{2}{-1} = 2$. Hence these curves are orthogonal.