1-1:50pm

Midterm Exam 1

Spring 2012 Math 0220

100 points total **Your name:** Solution

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

---

1. (10 points) Evaluate the limit, if it exists. If it does not exist explain why.

$$\lim_{x\to -5^-} \frac{3x+15}{|x+5|}$$

In your work mention what Rules, Laws, Theorems or Formulas you use.

Sln: $x \to -5^- \iff x \to -5$ and $x < -5$

$$|x+5| = \begin{cases} x+5, & \text{if } x+5 \geq 0 \\ -(x+5), & \text{if } x+5 < 0 \end{cases} = \begin{cases} x+5, & x \geq -5 \\ -(x+5), & x < -5 \end{cases}$$

Hence if $x < -5$, then $|x+5| = -(x+5)$.

$$\lim_{x\to -5^-} \frac{3x+15}{|x+5|} = \lim_{x\to -5^-} \frac{3(x+5)}{-(x+5)} = \lim_{x\to -5^-} (-3)$$

$$= \boxed{-3}$$

$$\left(\lim_{x\to a} c = c\right)$$

2. (a) (10 points) Find the functions $f \circ g$ and $g \circ f$ if

$$f(x) = 2 - \frac{1}{x} \quad \text{and} \quad g(x) = \frac{x-1}{x-2}$$

Simplify your answers.

$$f \circ g = f(g(x)) = 2 - \frac{1}{g(x)} = 2 - \frac{x-2}{x-1} = \frac{2x-2-x+2}{x-1}$$

$$\boxed{= \frac{x}{x-1}}$$

$$g \circ f = g(f(x)) = \frac{f(x)-1}{f(x)-2} = \frac{2-\frac{1}{x}-1}{2-\frac{1}{x}-2} = \frac{1-\frac{1}{x}}{-\frac{1}{x}} \cdot \frac{-x}{-x}$$

$$= \frac{-x+1}{1} = \boxed{-x+1}$$

(b) (5 points) Find the domain of the function $f \circ g$.

$f \circ g$ is defined whenever both $g(x)$ and $f(g(x))$ is defined (page 19).

Domain of $g(x)$: $D(g) = (-\infty, 2) \cup (2, \infty)$

Domain of $f(g(x))$: $D(f(g(x)) = (-\infty, 1) \cup (1, \infty)$

Therefore, $D(f \circ g) = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$

(If you didn't mention continuity of $f(x)$ on $[0,\pi]$ then your score ≤ 12.)

3. (15 points) Use the Intermediate Value Theorem to show that there is root of the equation $x^2 - 1 = \sin x$ in the interval $(0, \pi)$.

Rewrite the equation: $x^2-1-\sin x=0$

Let $f(x) = x^2-1-\sin x$.

Need to show, that there is c in $(0,\pi)$ s.t. $f(c)=0$.

$f(x)$ is continuous on $[0,\pi]$.

$f(0)=-1$, $f(\pi)=\pi^2-1-0=\pi^2-1>0$.

Take $N=0$. Then $f(0)<N<f(\pi)$ and the conditions of the IVT hold.

Then by the IVT there exists a number c in $(0,\pi)$ such that $f(c)=0$.

The last means $c^2-1-\sin c=0$

or $c^2-1=\sin c$. Hence c is a root of the equation $x^2-1=\sin x$ in the interval $(0,\pi)$.

4. (10 points) Find all horizontal and vertical asymptotes of the curve

$$y = \frac{\sqrt{4x^2+5}}{4x+5}$$

Justify your answer by calculating corresponding limits.

There is a potential v.a. where the denominator is 0, i.e. $4x+5=0$ or $x=-\frac{5}{4}$.

$$\lim_{x \to -\frac{5}{4}^+} \frac{\sqrt{4x^2+5}}{4x+5} = \infty \quad \text{because}$$

$\sqrt{4\cdot(-\frac{5}{4})^2+5} > 0$, $4x+5$ can be made arbitrary small and positive $\Rightarrow$ $\frac{\sqrt{4x^2+5}}{4x+5}$ can be made arbitrary large as $x \to -\frac{5}{4}^+$

Hence $\boxed{x=-\frac{5}{4}}$ is a v.a.

Now, $$\lim_{x\to\infty} \frac{\sqrt{4x^2+5}}{4x+5} = \lim_{x\to\infty} \frac{\sqrt{4x^2+5}\cdot\frac{1}{x}}{(4x+5)\cdot\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{4+\frac{5}{x^2}}}{4+\frac{5}{x}} = \frac{\sqrt{4}}{4} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+5}}{4x+5} = \lim_{x\to-\infty} \frac{\sqrt{4x^2+5}\cdot(\frac{1}{-x})}{(4x+5)\cdot\frac{1}{-x}} = \lim_{x\to-\infty} \frac{\sqrt{4+\frac{5}{x^2}}}{-4-\frac{5}{x}} = -\frac{1}{2}$$

since $\sqrt{x^2}=-x$ when $x<0$

$\boxed{\text{h.a.: } y=\frac{1}{2} \text{ and } y=-\frac{1}{2}}$

5. For the function $f(x) = 2\cos x - \sqrt{3x}$

(a) (5 points) find its first derivative $f'(x)$.

$$f(x) = 2\cos x - \sqrt{3} \cdot x^{1/2}$$

$$f'(x) = 2(-\sin x) - \sqrt{3} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$f'(x) = -2\sin x - \frac{\sqrt{3}}{2} x^{-\frac{1}{2}}$$

$$\text{or } f'(x) = -2\sin x - \frac{\sqrt{3}}{2\sqrt{x}}$$

$$f'(x) = -2\sin x - \frac{3}{2}(3x)^{-\frac{1}{2}}$$

(b) (5 points) find its second derivative $f''(x)$.

$$f''(x) = (f'(x))' = (-2\sin x)' - \frac{\sqrt{3}}{2}\left(x^{-\frac{1}{2}}\right)'$$

$$= -2\cos x - \frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) x^{-3/2}$$

$$f''(x) = -2\cos x + \frac{\sqrt{3}}{4} x^{-3/2}$$

$$\text{or } f''(x) = -2\cos x + \frac{\sqrt{3}}{4x\sqrt{x}} = -2\cos x + \frac{9}{4}(3x)^{-3/2}$$

6. (10 points) Find an equation of the tangent line to the curve

$$y = (2 + 3x)\cos x$$

at the point which x-coordinate is 0.

$x_1 = 0 \Rightarrow y_1 = (2+0)\cos 0 = 2$

$m = y'(x_1) = y'(0)$

$y' = 3\cos x + (2+3x)(-\sin x)$ (product rule)

$y'(0) = 3 + (2+0)(0) = 3 = m$

tan. line equation:

$y = 2 + 3(x - 0)$

$\boxed{y = 3x + 2}$

7. (15 points) Using any method find the derivative $G'(\theta)$ of the function

$$G(\theta) = \sqrt{\sin(\cos^2\theta)}$$

In your work mention what Rules, Laws, Theorems or Formulas you use.

$$G'(\theta) = \frac{1}{2\sqrt{\sin(\cos^2\theta)}} \cdot \cos(\cos^2\theta) \cdot 2\cos\theta(-\sin\theta)$$

$$G'(\theta) = -\frac{\cos\theta \cdot \sin\theta \cdot \cos(\cos^2\theta)}{\sqrt{\sin(\cos^2\theta)}}$$

8. (15 points) Use implicit differentiation to find an equation of the tangent line to the curve $2\cos x \sin y = 1$ at the point $(\pi/4, \pi/4)$.

Implicit differentiation:

$\frac{d}{dx}[2\cos x \sin y] = \frac{d}{dx}[1]$ Product, Chain Rules

$2(-\sin x)\sin y + 2\cos x \cos y \cdot y' = 0$

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Plug in $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$ into the equation:

$2\left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} y' = 0$

$-1 + y' = 0$, $y' = 1 = m$

Tan. line equation:

$y = \frac{\pi}{4} + 1 \cdot \left(x - \frac{\pi}{4}\right)$

$\boxed{y = x}$

bonus problem [8 points extra] Consider the functions

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x < 0 \\ x+1 & \text{if} \quad x \geq 0 \end{cases} \quad \text{and} \quad g(x) = b^2x^2 + bx + b^2$$

Find all values of b such that $g(f(x))$ is continuous everywhere.

f is made of polynomials, g is a polynomial. The only discontinuity can appear at $x = 0$.

Continuity condition at $x = 0$ is

$$\lim_{x \to 0^-} g(f(x)) = \lim_{x \to 0^+} g(f(x)) = g(f(0))$$

$$\lim_{x \to 0^-} g(f(x)) = g\left(\lim_{x \to 0^-} f(x)\right) = g(0) = b^2$$

$$\lim_{x \to 0^+} g(f(x)) = g\left(\lim_{x \to 0^+} f(x)\right) = g(1) = 2b^2 + b$$

$$g(f(0)) = g(1) = 2b^2 + b.$$

Hence $b^2 = 2b^2 + b$ must hold to make $g(f(x))$ continuous everywhere.

$$b^2 = 2b^2 + b \iff b^2 + b = 0 \iff b(b+1) = 0$$

$$\iff \boxed{b = 0 \text{ or } b = -1}$$