9-9:50am

Midterm Exam 1

Spring 2012 Math 0220

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

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1. (10 points) Find the domain of the function

$$g(t) = \frac{\sqrt{1-t^2}}{2\sin t - 1}$$

Solution: $1-t^2 \geq 0$ and $2\sin t - 1 \neq 0$

$1-t^2 \geq 0 \iff (1-t)(1+t) \geq 0 \iff t \in [-1,1]$

$2\sin t - 1 \neq 0 \iff \sin t \neq \frac{1}{2} \iff t \neq \frac{\pi}{6}$ inside $[-1,1]$

$D = [-1, \frac{\pi}{6}) \cup (\frac{\pi}{6}, 1]$

(or all reals inside the interval $[-1,1]$ except $\frac{\pi}{6}$)

You also can write:

$D = \{t \in \mathbb{R} \mid t \in [-1, \frac{\pi}{6}) \cup (\frac{\pi}{6}, 1]\}$

2. (15 points) Evaluate the limit, if it exists. If it does not exist explain why.

$$\lim_{x\to 0} x^3 \sin\left(\frac{3}{x}\right)$$

In your work mention what Rules, Laws, Theorems or Formulas you use.

Sln By the property of sin x we have:

$$-1 \le \sin\left(\frac{3}{x}\right) \le 1$$

Then we multiply all parts in the inequality by x^3.

If $x>0$, then $-x^3 \le x^3 \sin\left(\frac{3}{x}\right) \le x^3$

If $x<0$, then $-x^3 \ge x^3 \sin\left(\frac{3}{x}\right) \ge x^3$

$\lim\limits_{x\to 0} x^3 = 0$. Applying the Squeeze Thm to both inequalities we get

$$\lim_{x\to 0} x^3 \sin\left(\frac{3}{x}\right) = 0. \text{ (the limit exists)}$$

Note: $\lim\limits_{x\to 0} x^3 \sin\left(\frac{3}{x}\right) = \lim\limits_{x\to 0} x^3 \cdot \lim\limits_{x\to 0} \sin\left(\frac{3}{x}\right) = 0 \cdot \lim\limits_{x\to 0} \sin\left(\frac{3}{x}\right) = 0$ is a wrong way.

or else $\lim\limits_{x\to 0} \frac{1}{x} = \lim\limits_{x\to 0} x \cdot \frac{1}{x^2} = \lim\limits_{x\to 0} x \cdot \lim\limits_{x\to 0} \frac{1}{x^2} = 0 \cdot \lim\limits_{x\to 0} \frac{1}{x^2} = 0$ No!

3. (10 points) Define if the function

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if} \quad x < 2 \\ 2x - 1 & \text{if} \quad x \geq 2 \end{cases}$$

is continuous or discontinuous at the point $a = 2$. Support your answer by using the definition of continuity. No credit will be given if you do not use the definition in your proof. Do not draw a graph of the function.

Sln: $f(a) = f(2) = 2 \cdot 2 - 1 = 3$.

$f(x)$ is defined at 2.

-1 if this part is omit

$$\boxed{\lim_{x \to 2^-} f(x)} = \lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2^-} \frac{(x-2)(x+1)}{x-2}$$

$$= \lim_{x \to 2^-} (x+1) \underset{DSP}{=} 2 + 1 = 3$$

-1 if omit

$$\boxed{\lim_{x \to 2^+} f(x)} = \lim_{x \to 2^+} (2x - 1) \underset{DSP}{=} 2 \cdot 2 - 1 = 3$$

We have

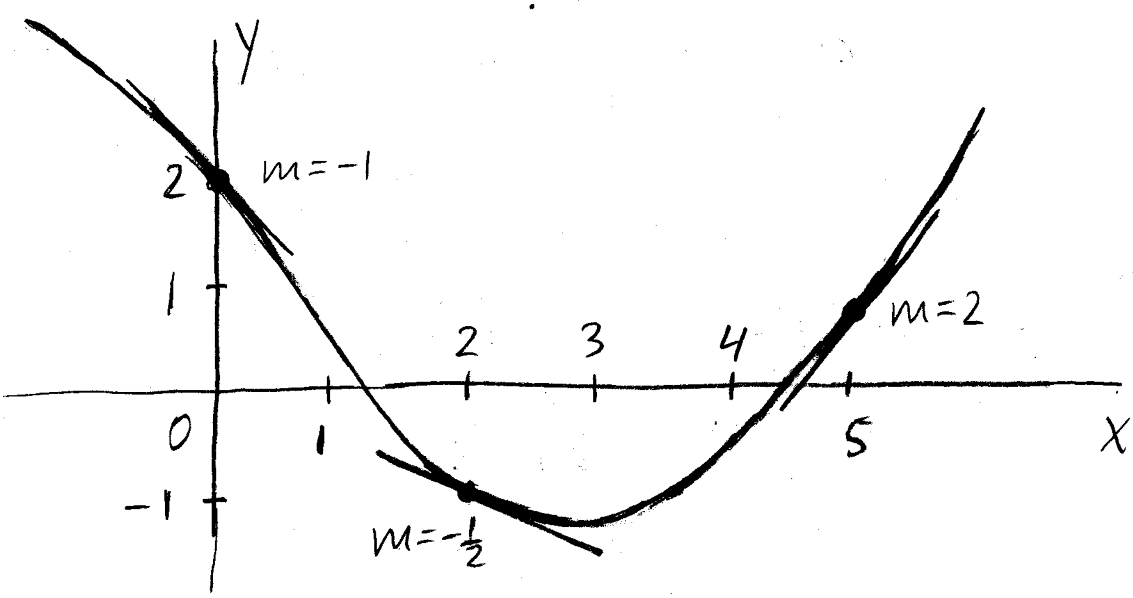
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

Hence $\lim_{x \to 2} f(x) = f(2)$ and $f(x)$ is continuous at 2 by the definition of continuity.

4. (10 points) Sketch the graph of an example of a function $g(x)$ if it satisfies all the given conditions

$$g(0) = 2, \quad g'(0) = -1, \quad g(2) = -1, \quad g'(2) = -\frac{1}{2}, \quad \text{and} \quad g'(5) = 2$$

Mark all the essential points on the axes.



(tangent lines at $x=0$, $x=2$, and $x=5$ are shown)

Each condition if it is drawn right gives 2 points

5. A particle moves according to a law of motion $s(t) = t^3 - 12t^2 + 36t$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) (5 points) What is its velocity after 3 seconds?

(b) (5 points) When is the particle at rest?

(c) (5 points) Find the total distance traveled during the first 8 seconds.

Sln: (a) $v(t) = \frac{ds}{dt} = 3t^2 - 24t + 36$

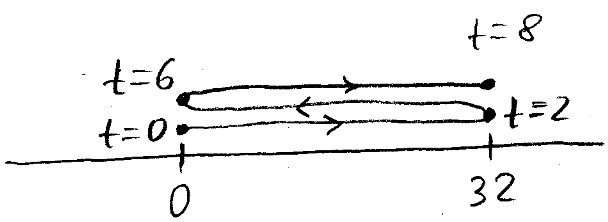
$$v(t) = 3(t^2 - 8t + 12) = 3(t-2)(t-6)$$

$$v(3) = 3 \cdot 1 \cdot (-3) = \boxed{-9 \quad ft/sec}$$

(b) $v(t) = 0 \iff 3(t-2)(t-6) = 0 \iff$ $t = \boxed{2 \text{ sec}}$, $t = \boxed{6 \text{ sec}}$

turning points

(c)



$$s(t) = t(t^2 - 12t + 36) = t(t-6)^2$$

$$s(0) = 0, \quad s(2) = 2(-4)^2 = 32, \quad s(6) = 0$$

$$s(8) = 8 \cdot 2^2 = 32$$

From the graph: $d = 3 \cdot 32 = \boxed{96 \text{ ft}}$

6. (10 points) Using any method find the derivative $f'(x)$ of the function

$$f(x) = \frac{\sqrt{2x}+2}{\sqrt{2x}-2}$$

Simplify your answer. In your work mention what Rules, Laws, Theorems or Formulas you use.

Solution: $(\sqrt{2x})' = (\sqrt{2}\cdot\sqrt{x})' = \sqrt{2}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2x}}$

Quotient Rule:

$$f'(x) = \frac{\frac{1}{\sqrt{2x}}(\sqrt{2x}-2) - \frac{1}{\sqrt{2x}}(\sqrt{2x}+2)}{(\sqrt{2x}-2)^2}$$

$$f'(x) = \frac{\sqrt{2x}-2-\sqrt{2x}-2}{\sqrt{2x}\,(\sqrt{2}(\sqrt{x}-\sqrt{2}))^2} = \frac{-4}{\sqrt{2}\cdot\sqrt{x}\cdot 2(\sqrt{x}-\sqrt{2})^2}$$

$$f'(x) = -\frac{\sqrt{2}}{\sqrt{x}\,(\sqrt{x}-\sqrt{2})^2} \quad \text{or} \quad f'(x) = -\frac{4}{\sqrt{2x}\,(\sqrt{2x}-2)^2}$$

Another way: $f(x) = \frac{\sqrt{2x}-2+4}{\sqrt{2x}-2} = 1+4\cdot\frac{1}{\sqrt{2x}-2}$

$$f'(x) = \underset{\text{Quotient Rule}}{4\cdot\frac{0-\frac{1}{\sqrt{2x}}}{(\sqrt{2x}-2)^2}} = -\frac{4}{\sqrt{2x}\,(\sqrt{2x}-2)^2}$$

7. (15 points) Find the first and second derivatives of the function

$$g(x) = \sqrt{2x^2 - 1}$$

Simplify your answer. In your work mention what Rules, Laws, Theorems or Formulas you use.

$$g(x) = (2x^2-1)^{1/2}$$

Power, Chain Rules

$$g'(x) = \frac{1}{2}(2x^2-1)^{-\frac{1}{2}}(4x)$$

$$g'(x) = 2x \cdot (2x^2-1)^{-\frac{1}{2}}$$

(9 points)

Product, Power chain Rules

$$g''(x) = 2(2x^2-1)^{-\frac{1}{2}} + 2x\left(-\frac{1}{2}\right)(2x^2-1)^{-3/2}(4x)$$

$$g''(x) = \frac{2}{\sqrt{2x^2-1}} - \frac{4x^2}{\left(\sqrt{2x^2-1}\right)^3}$$

(15 points)

$$\left[\text{or } g''(x) = \frac{4x^2-2-4x^2}{\left(\sqrt{2x^2-1}\right)^3} = -\frac{2}{(2x^2-1)^{3/2}} = -2(2x^2-1)^{-3/2} \right]$$

8. (15 points) Use implicit differentiation to find an equation of the tangent line to the curve $x^{2/3} + y^{2/3} = 5$ at the point $(8, 1)$.

Sln Differentiate both sides w.r.t. x to find $y' = \frac{dy}{dx}$:

$$\frac{d}{dx}\left[x^{2/3} + y^{2/3}\right] = \frac{d}{dx}[5]$$

$$\cancel{\tfrac{2}{3}}x^{-1/3} + \cancel{\tfrac{2}{3}}y^{-1/3}\,y' = 0$$

$$(8)^{-1/3} + (1)^{-1/3}\,y' = 0\ , \quad \frac{1}{2} + y' = 0$$

$$m = y' = -\frac{1}{2}$$

tan. line: $\quad y = 1 + \left(-\frac{1}{2}\right)(x-8)$

$$y = -\frac{1}{2}x + 5$$

bonus problem [8 points extra] Use the definition of the derivative to find the derivative of the function $f(x) = g(cx)$, where $g(x)$ is a differentiable function and c is a constant. No credit be given if the derivative will be found without using the definition.

Definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{g(c(x+h)) - g(cx)}{h}$$

$c(x+h) = cx + ch$. Denote $y = cx$. Then

$$f'(x) = \lim_{h \to 0} \frac{g(y+ch) - g(y)}{h} = \lim_{h \to 0} \frac{c(g(y+ch) - g(y))}{ch}$$

$$= c \lim_{h_c \to 0} \frac{g(y+h_c) - g(y)}{h_c} = c \frac{dg}{dy}$$

where $h_c = ch$

Hence $\boxed{\frac{df}{dx} = c \frac{dg}{dy}, \text{ where } y = cx}$

This equality corresponds to the Chain Rule, with g as an outer and $y = cx$ as an inner functions.