10-10:50am

Midterm Exam 2

Spring 2012 Math 0220

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

---

1. (15 points) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a hemispherical dome with diameter 120 cm.

$V(r) = \frac{1}{2} \cdot \frac{4}{3} \pi r^3 = \frac{2}{3} \pi r^3$, $r = \frac{120\text{cm}}{2} = 60\text{ cm}$

We need to find dV to estimate the amount (which is volume) of paint needed:

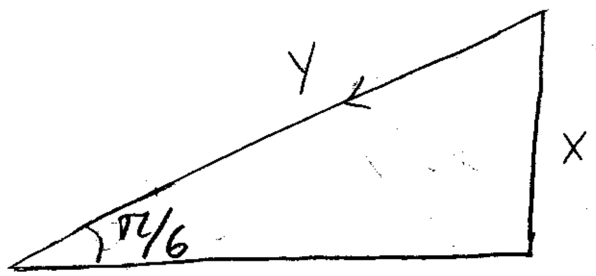
$dV = V'(r)\,dr$, $V'(r) = 2\pi r^2$, $dr \approx 0.02\text{cm}$

$dV \approx 2\pi (60\text{ cm})^2 (0.02\text{cm})$

$dV \approx 2\pi \cdot 36 \cdot 100 \cdot \frac{2}{100}\text{ cm}^3$

$dV \approx 4\pi \cdot 36\text{ cm}^3 = \boxed{144\pi\text{ cm}^3}$

2. (15 points) A kite is flying at an angle of elevation of $\pi/6$. The kite string is being taken in at the rate of 2 foot per second. If the angle of elevation does not change, how fast is the kite losing altitude?



length of the string is Y

elevation is X

$$\sin\frac{\pi}{6} = \frac{X}{Y} \quad \text{or} \quad \frac{X}{Y} = \frac{1}{2}$$

$$\text{or} \quad X = \frac{1}{2}Y \Rightarrow \frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}\cdot 2 \ \text{ft/sec}$$

$$\frac{dx}{dt} = 1 \ \text{ft/sec}$$

Answer: the kite is losing altitude with the rate of 1 foot/sec.

3. (15 points) Find the limit

$$\lim_{x\to\pi} \frac{\tan^{-1} x}{\tan^{-1} 2x}$$

Do not simplify your answer.

$$\lim_{x\to\pi} \frac{\tan^{-1} x}{\tan^{-1} 2x} \underset{DSP}{=} \frac{\tan^{-1} \pi}{\tan^{-1} 2\pi}$$

4. (10 points) Suppose that $f(0) = -3$ and $f'(x) = 5$ for all values of x. Use the Mean Value Theorem to show that $f(x) = 5x - 3$.

$f'(x)=5$ for all x means that $f(x)$ is differentiable on $(-\infty,\infty)$ and hence it is also continuous.

By MVT for arbitrary a and b there is $c \in (a,b)$ such that

$$f(b)-f(a) = f'(c)(b-a) = 5(b-a)$$

since $f'(x) = 5$ for all x and for $x=c$ in particular. Since a and b are arbitrary take $a=0$, $b=x$ to get

$f(x)-f(0) = 5(x-0)$ or $f(x)+3 = 5x$,

which gives $\boxed{f(x)= 5x-3}$

5. (15 points) Find the absolute maximum and absolute minimum values of the function

$$f(x) = \frac{x}{x^2+4} \quad \text{on the interval} \quad [-4, 1]$$

$$f'(x) = \frac{(x)'(x^2+4) - x(x^2+4)'}{(x^2+4)^2}$$ (Quotient Rule)

$$f'(x) = \frac{x^2+4-x\cdot 2x}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

CP's : $4 - x^2 = 0 \iff x = -2,\ x = 2$

$f'(x)$ exists everywhere

$$f(-4) = \frac{-4}{20} = -\frac{1}{5}$$

$$f(-2) = \frac{-2}{8} = -\frac{1}{4} \qquad \left(-\frac{1}{4} < -\frac{1}{5}\right)$$

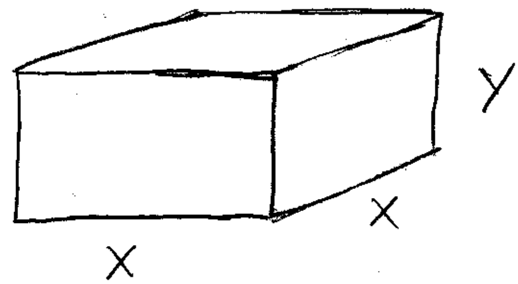
$$f(1) = \frac{1}{5}$$, 2 is not in $[-4, 1]$

Abs. max. value is $\frac{1}{5}$

Abs. min value is $-\frac{1}{4}$

13 points if you did not show that $4000\ cm^3$ is the largest (abs. max) volume

6. (15 points) If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.



Material = surface area A

$$A = 4xy + x^2 = 1200$$

$$V = x^2 y \qquad y = \frac{1200 - x^2}{4x}$$

$$V(x) = x^2 \cdot \frac{1200 - x^2}{4x} = 300x - \frac{1}{4}x^3 \to \max$$

$$0 < x < \infty$$

$$V'(x) = 300 - \frac{3}{4}x^2 \qquad \text{CP's: } 300 - \frac{3}{4}x^2 = 0$$

$$x^2 = 400, \quad x = 20\ \text{cm} \quad (\text{one point!})$$

$$V''(x) = -\frac{3}{2}x < 0 \quad \text{since } x > 0$$

By the 2^{nd} Der. Test $V(x)$ has loc. max which is the abs max at $x = 20$

$$V(20) = 300 \cdot 20 - \frac{1}{4}20^3 = 20\left(300 - \frac{400}{4}\right)$$

$$= 20 \cdot 200 = 4{,}000\ cm^3$$

Answer: $4{,}000\ cm^3$

7. (15 points) Use Newton's method to approximate the number $\sqrt[5]{100}$. Take $x_1 = 2$ as the initial approximation and find the second approximation x_2 to the result.

Let $\sqrt[5]{100} = x \iff x^5 = 100 \iff x^5 - 100 = 0$

Let $f(x) = x^5 - 100$, then $f'(x) = 5x^4$

NM: $x_{n+1} = x_n - \dfrac{x_n^5 - 100}{5x_n^4}$

$x_1 = 2$

$$x_2 = 2 - \frac{2^5 - 100}{5 \cdot 2^4} = 2 - \frac{32 - 100}{5 \cdot 16}$$

$$x_2 = 2 - \frac{-68}{80} = 2 + \frac{17}{20} = 2\frac{17}{20} = 2.85$$

$\boxed{x_2 = 2\frac{17}{20}}$ or $\boxed{x_2 = 2.85}$

bonus problem [10 points extra] Evaluate the limit $\lim_{x \to a} \frac{x^n - a^n}{x^{1/n} - a^{1/n}}$.

$$\lim_{x \to a} \frac{x^n - a^n}{x^{\frac{1}{n}} - a^{\frac{1}{n}}} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{x \to a} \frac{n x^{n-1}}{\frac{1}{n} x^{\frac{1}{n}-1}}$$

$$= \lim_{x \to a} n^2 x^{n-\frac{1}{n}} \underset{DSP}{=} \boxed{n^2 a^{n-\frac{1}{n}}}$$