1-1:50pm

Midterm Exam 2

Spring 2012 Math 0220

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

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1. (15 points) Find $\left(f^{-1}\right)'(2)$ if $f(x) = \dfrac{1}{x-1}$, $x > 1$.

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

$$f^{-1}(2) = x \iff f(x) = 2 \iff \frac{1}{x-1} = 2$$

$$x = \frac{3}{2}$$

Hence $f^{-1}(2) = \frac{3}{2}$

$$f'(x) = \left((x-1)^{-1}\right)' = -(x-1)^{-2} \cdot 1 \quad \text{(Chain Rule)}$$

$$f'\left(\tfrac{3}{2}\right) = -\left(\tfrac{3}{2} - 1\right)^{-2} = -\left(\tfrac{1}{2}\right)^{-2} = -2^2 = -4$$

Hence $(f^{-1})'(2) = \dfrac{1}{-4} = \boxed{-\dfrac{1}{4}}$

$$\left(f^{-1}(x) = x^{-1} + 1,\ (f^{-1})'(x) = -x^{-2},\ (f^{-1})'(2) = -\frac{1}{4} \right)$$

Another way: $e^{5800k} = \frac{1}{2} \Leftrightarrow e^{k} = \left(\frac{1}{2}\right)^{1/5800}$
$\Leftrightarrow e^{kt} = \left(\frac{1}{2}\right)^{t/5800}$

2. (15 points) A piece of cloth is recovered by some archaeologists. Upon examination, it is found that 75% of the original Carbon 14 remains in the cloth. If the half-life of Carbon 14 is 5800 years, calculate the age of the piece of cloth.

Let $m(t)$ denote the mass of Carbon 14 after t years. Then

$m(t) = m(0)e^{kt}$.

$m(5800) = \frac{1}{2}m(0) \Leftrightarrow m(0)e^{5800k} = \frac{1}{2}m(0)$

$\Leftrightarrow e^{5800k} = \frac{1}{2} \Leftrightarrow 5800k = \ln\left(\frac{1}{2}\right)$

$\Leftrightarrow k = \ln\left(\frac{1}{2}\right)^{1/5800} \Rightarrow m(t) = m(0)\left(\frac{1}{2}\right)^{t/5800}$

$75\% = \frac{3}{4}$. Then $m(t) = \frac{3}{4}m(0)$. Find t.

$m(0)\left(\frac{1}{2}\right)^{t/5800} = \frac{3}{4}m(0) \Leftrightarrow \left(\frac{1}{2}\right)^{t/5800} = \frac{3}{4}$.

$\frac{t}{5800}\ln\left(\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right) \Leftrightarrow \boxed{t = 5800 \cdot \frac{\ln(3/4)}{\ln(1/2)} \text{ years}}$

Possible answers: $t = 5800\,\frac{\ln(.75)}{\ln(.5)} = 5800\log_{\frac{1}{2}}\left(\frac{3}{4}\right)$

$t = 5800 \cdot \frac{\ln 4 - \ln 3}{\ln 2} = 5800 \cdot \log_2\left(\frac{4}{3}\right) = 5800(2 - \log_2 3)$

3. (15 points) Find the limit

$$\lim_{x\to\pi} \frac{\cos^3(x/2)}{\sin x}$$

$$\lim_{x\to\pi} \frac{\cos^3(x/2)}{\sin x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{x\to\pi} \frac{3\cos^2(x/2)(-\sin x/2)(\frac{1}{2})}{\cos x}$$

$$\underset{DSP}{=} \frac{3\cdot 0^2(-1)\cdot\frac{1}{2}}{-1} = \frac{0}{-1} = 0$$

4. (15 points) Find the absolute maximum and absolute minimum values of the function

$$f(x) = xe^{-x^2/2} \quad \text{on the interval} \quad [-1, 4]$$

$$f'(x) = e^{-x^2/2} + x e^{-x^2/2}\left(-\frac{2x}{2}\right)$$

(Product + Chain Rules)

CP's: $f'(x) = e^{-x^2/2}(1 - x^2) = 0 \iff 1 - x^2 = 0$

since $e^{-x^2/2} > 0$ for all x.

$f'(x)$ exists everywhere.

CP's are $x = -1$, $x = 1$

$$f(-1) = -e^{-1/2} = -\frac{1}{\sqrt{e}}$$

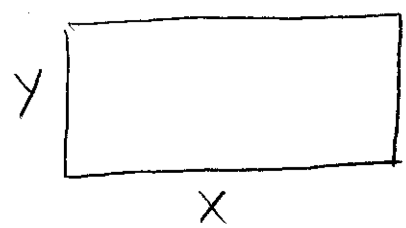
$$f(1) = e^{-1/2} = \frac{1}{\sqrt{e}}$$

$$f(4) = 4e^{-8} = \frac{4}{e^8} \quad \left(\frac{1}{\sqrt{e}} > \frac{4}{e^8}\right)$$

Abs. min. value is $-\frac{1}{\sqrt{e}}$

Abs. max. value is $\frac{1}{\sqrt{e}}$

5. (15 points) Find the dimensions of a rectangle with area 400 m^2 whose perimeter as small as possible. Support your answer.



$A = xy$, $xy = 400$, $y = \frac{400}{x}$

perimeter is $P = 2x + 2y$

$P(x) = 2x + 2 \cdot \frac{400}{x} = 2x + 800x^{-1} \to \min$

$0 < x < \infty$

$P'(x) = 2 - 800x^{-2} = 0 \iff 1 - \frac{400}{x^2} = 0$

CP's: $\frac{x^2 - 400}{x^2} = 0$, $x^2 = 400$, $x = 20$ $(x > 0)$

$P'(x)$ exists for all $x > 0$

The only CP is at $x = 20$

$P''(x) = 1600x^{-3} > 0$ when $x > 0$. By the 2^{nd} Der. Test $P(x)$ attains its loc. min which is the abs. min at $x = 20$; $y = 20$

Answer: $20m \times 20m$ (a square).

6. (10 points) Find the limit

$$\lim_{x\to\infty} \left(2^{-x} \sin 3x\right)$$

$$-1 \le \sin 3x \le 1$$

$$-\frac{1}{2^x} \le \frac{\sin 3x}{2^x} \le \frac{1}{2^x}$$

$$\lim_{x\to\infty}\left(-\frac{1}{2^x}\right) = 0, \quad \lim_{x\to\infty}\frac{1}{2^x} = 0$$

By the Squeeze Thm

$$\lim_{x\to\infty}\frac{\sin 3x}{2^x} = 0 \quad \text{or} \quad \lim_{x\to\infty}\left(2^{-x}\sin 3x\right) \boxed{= 0}$$

7. (15 points) For the equation $x^2 = 2$ use Newton's method with the initial approximation $x_1 = 2$ to find the third approximation x_3 to the positive root.

$$X^2 = 2 \iff f(X) = X^2 - 2 = 0$$

$$f'(X) = 2X.$$

$$\underline{NM}: \quad X_{n+1} = X_n - \frac{X_n^2 - 2}{2X_n} = \frac{X_n^2 + 2}{2X_n}$$

$$X_{n+1} = \frac{1}{2} X_n + X_n^{-1}$$

$$X_1 = 2$$

$$X_2 = \frac{1}{2} \cdot 2 + 2^{-1} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$X_3 = \frac{1}{2} \cdot \frac{3}{2} + \frac{2}{3} = \frac{3}{4} + \frac{2}{3} = \boxed{\frac{17}{12} = 1\frac{5}{12}}$$

bonus problem [10 points extra] Show that the equation $2\cos x - 3x = 0$ has exactly one real root.

Let $f(x) = 2\cos x - 3x$.

$f(x)$ is contin and dif-le on $(-\infty, \infty)$.

$f(0) = 2 > 0$, $f(\frac{\pi}{2}) = 0 - \frac{3\pi}{2} < 0$

By IVT there is one root of $f(x)$ in $(0, \frac{\pi}{2})$ $\Rightarrow$ there is one root of the equation.

Assume there are two roots a and b

Then $f(a) = f(b)$. By Rolle's Thm there is c in (a, b) s.t. $f'(c) = 0$.

On the other hand $f'(x) = -2\sin x - 3$ $\leq 2 - 3 \leq -1$ and $f'(x)$ is never 0.

A contradiction! Hence the assumption was wrong and the equation cannot have more than one root.