9-9:50am

Midterm Exam 2

Spring 2012 Math 0220

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

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1. (10 points) Find a formula for the inverse of the function

$$f(x) = \frac{1-e^x}{1+e^x} \quad \text{when} \quad -1 < x < 1$$

$$x = \frac{1-e^y}{1+e^y}, \qquad x(1+e^y) = 1-e^y,$$

$$xe^y + e^y = 1 - x, \qquad e^y = \frac{1-x}{1+x}$$

$$y = \ln\left(\frac{1-x}{1+x}\right)$$

$$f^{-1}(x) = \ln\left(\frac{1-x}{1+x}\right)$$

✓

2. (15 points) Suppose that the population of a colony of bacteria increases exponentially. At the start of an experiment, there are 6,000 bacteria, and one hour later, the population has increased to 6,400. How long will it take for the population to reach 10,000?

Let $P(t)$ be the population. Then

$P(t) = P_0 e^{kt}$ with $P_0 = 6000$.

$P(1) = 6000 e^{k \cdot 1} = 6400 \Rightarrow e^k = \frac{6400}{6000}$

$\Rightarrow e^k = \frac{16}{15} \Rightarrow k = \ln \frac{16}{15}$.

Hence, $P(t) = 6000 e^{\left(\ln \frac{16}{15}\right) t}$

or $P(t) = 6000 \left(\frac{16}{15}\right)^t$

Now, $P(t) = 6000 \left(\frac{16}{15}\right)^t = 10000$

$\Rightarrow \left(\frac{16}{15}\right)^t = \frac{10}{6} = \frac{5}{3} \Rightarrow t \ln \frac{16}{15} = \ln \frac{5}{3}$

$\boxed{t = \frac{\ln 5 - \ln 3}{\ln 16 - \ln 15} \text{ hrs}}$ or $\boxed{t = \frac{\ln 5/3}{\ln 16/15} \text{ hrs}}$

3. (15 points) Find the derivative of the function and simplify it if possible

$$f(x) = x\ \cos^{-1}\left(e^{3x}\right)$$

$$f'(x) = (x)'\cos^{-1}(e^{3x}) + x\left(\cos^{-1}(e^{3x})\right)'$$

(Product Rule)

$$f'(x) = \cos^{-1}(e^{3x}) + x\left(-\frac{1}{\sqrt{1-(e^{3x})^2}}\right)e^{3x}\cdot 3$$

(Chain Rule)

$$f'(x) = \cos^{-1}(e^{3x}) - \frac{3xe^{3x}}{\sqrt{1-e^{6x}}}$$

4. (15 points) Find the limit

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$$

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{x\to 0} \frac{e^x - 1}{2x}$$

$$\underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{x\to 0} \frac{e^x}{2} \underset{\text{DSP}}{=} \frac{e^0}{2} = \boxed{\frac{1}{2}}$$

5. For the function $f(x) = xe^x$ Domain $D = (-\infty, \infty)$

(a) (5 points) Find the intervals on which f is increasing or decreasing.

(b) (5 points) Find the local maximum and local minimum values of f.

(c) (5 points) Find the intervals of concavity (where f is CU and CD) and the inflection points.

(a) $f'(x) = e^x + xe^x = (x+1)e^x$, $e^x > 0$ for all x

$f'(x) > 0 \iff x+1 > 0 \iff x > -1$

$f'(x) < 0 \iff x < -1$

	$x < -1$	-1	$x > -1$
f'	$-$		$+$
f	$\searrow$		$\nearrow$

(b) $f'(x) < 0$ when $x < -1$ and $f'(x) > 0$ when $x > -1$. By the 1st Der Test $f(x)$ has a loc. min at $x = -1$,

$f(-1) = -e^{-1} = -\frac{1}{e}$ is a loc. min. value

There is no loc. max's.

(c) $f''(x) = e^x + (x+1)e^x = (x+2)e^x$.

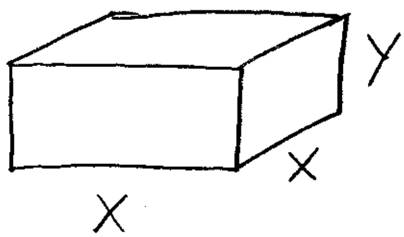
$f''(x) = 0 \iff x+2 = 0 \iff x = -2$

IP: $(-2, f(-2)) = (-2, -\frac{2}{e^2})$

$f''(x) > 0 \iff x > -2$
f is CU

$f''(x) < 0 \iff x < -2$
f is CD

6. (15 points) A box with a square base and open top must have a volume of 8000 cm^2. Find the dimensions of the box that minimize the amount of material used.



Volume $V = x^2 y$.

Let A denote the surface area which equals to the amount of material used.

$A = x^2 + 4xy$, where $0 < x < \infty$.

$y = \frac{V}{x^2} = \frac{8000}{x^2} \Rightarrow A(x) = x^2 + 32{,}000\, x^{-1}$.

$A(x) \to \min$. $A'(x) = 2x - 32{,}000\, x^{-2}$

CP: $2x^3 - 32{,}000 = 0 \iff x = \sqrt[3]{16{,}000}$,

$x = 10\sqrt[3]{16} = 10\sqrt[3]{8 \cdot 2} = 20\sqrt[3]{2}$ (the only CP).

$A'(x)$ exists everywhere in $(0, \infty)$ (domain of A)

$A''(x) = 2 + 64{,}000\, x^{-3} > 0$ everywhere in $(0, \infty)$

Hence by the 2^{nd} Der. Test A has a loc. min which is the abs. min.

at $x = 20\sqrt[3]{2}$

$y = \frac{8000}{400(2)^{2/3}} = \frac{10 \cdot 2}{2^{2/3}} = 10\sqrt[3]{2}$

Answer: the side of the base is $20\sqrt[3]{2}$ cm, the height is $10\sqrt[3]{2}$ cm.

7. (15 points) For the equation $x^3 - 2x - 5 = 0$ use Newton's method with the initial approximation $x_1 = 2$ to find the second approximation x_2 to the positive root.

Let $f(x) = x^3 - 2x - 5 \Rightarrow f'(x) = 3x^2 - 2$

NM: $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$

$x_1 = 2$

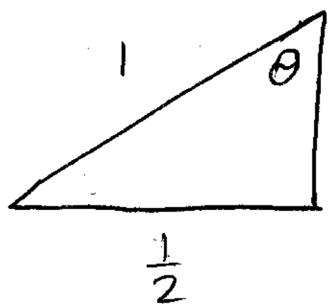
$x_2 = 2 - \dfrac{2^3 - 2\cdot 2 - 5}{3\cdot 2^2 - 2} = 2 - \dfrac{8 - 4 - 5}{12 - 2}$

$x_2 = 2 - \dfrac{-1}{10} = 2 + \dfrac{1}{10}$

$\boxed{x_2 = 2.1}$

bonus problem [10 points extra] Find the exact value of the expression $2^{3\log_2 3} + \tan(\sin^{-1}\frac{1}{2})$.

$$2^{3\log_2 3} = 2^{\log_2 3^3} = 3^3 = 27$$



$$\sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

From the picture $\sin\theta = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$

Hence $\tan\left(\sin^{-1}\frac{1}{2}\right) = \tan\theta = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{\sqrt{3}}{3}$

Answer: $\boxed{27 + \frac{\sqrt{3}}{3}}$ $\boxed{= 27 + \frac{1}{\sqrt{3}}}$

Another way: $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$