9am

Quiz 2

Spring 2012 Your name: Solutions

Math 0220 Your TA's name: ____________

No calculators. Show all your work (no work = no credit). Write neatly.

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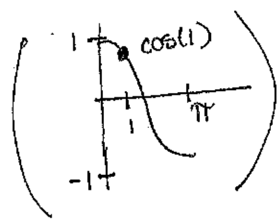
1. [5 points] Use the Intermediate Value Theorem to show that there is root of the equation $x = \cos x$ in the interval $(0, 1)$. [Hint: To apply IVT first check that the corresponding function satisfies all the conditions of the theorem].

1. Let $f(x) = x - \cos x$. We will use IVT to show there is $c \in (0,1)$ such that $f(c) = 0$, i.e. $c = \cos(c)$.

2. f is continuous as a difference of two continuous functions, x & $\cos x$.

3. $f(0) = 0 - \cos(0) = 0 - 1 = -1 < 0$
   $f(1) = 1 - \cos(1) > 0$ since $\cos(1) \in (0,1)$



4. By IVT, since f is continuous & $f(0) < 0 < f(1)$, there is $c \in (0,1)$ such that $f(c) = 0$.

2. [5 points] Find an equation of the tangent line to the curve $\overset{f(x)}{y} = \sqrt{x}$ at the point $(1, 1)$. To compute the slope use the definition of the derivative.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1}$$

LL + DSP

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \longrightarrow \text{slope of tangent line to } f \text{ at } x=1$$

OR

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1} \underset{\substack{\uparrow \\ LL + DSP}}{=} \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

$(1,1)$ is a point on the tangent line, $m = \frac{1}{2}$, so...

$$y - 1 = \tfrac{1}{2}(x-1)$$

$$y = \tfrac{1}{2}x + \tfrac{1}{2}$$

3. [5 points] Find all horizontal and vertical asymptotes of the curve

$$y = \frac{x^2+x-2}{x^2-1} = \frac{(x+2)(x-1)}{(x+1)(x-1)}$$

1. HAs

$$\lim_{x\to\infty} \frac{x^2+x-2}{x^2-1}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1+\frac{1}{x}+2\left(\frac{1}{x^2}\right)}{1-\frac{1}{x^2}}$$

$$\underset{\text{DSP}}{\overset{\uparrow}{=}} \frac{1+0+2(0)}{1+0} = 1 \Rightarrow \text{HA at } y=1$$

$\left(\lim_{x\to-\infty}\frac{x^2+x-2}{x^2-1} = 1 \text{ by same process}\right)$

2. VAs: Roots of denominator are $x=-1$ & $x=1$. When $x\neq 1$, $y=\frac{x+2}{x+1}$

$$\lim_{\substack{x\to 1\\(x\neq 1)}}\frac{x^2+x-2}{x^2-1} = \lim_{\substack{x\to 1\\(x\neq 1)}}\frac{x+2}{x+1}\underset{\text{DSP}}{\overset{\uparrow}{=}}\frac{1+2}{1+1}=\frac{3}{2}\Rightarrow x=1 \text{ is a hole, not a VA}$$

$$\lim_{x\to-1^-}\frac{x^2+x-2}{x^2-1} = \lim_{x\to-1^-}\frac{x+2}{x+1} = \text{"}\frac{1}{0^-}\text{"} = -\infty \quad \left(\begin{matrix}x+2\to 1\\ x+1\to 0^-\end{matrix}\right)$$

$$\lim_{x\to-1^+}\frac{x^2+x-2}{x^2-1} = \lim_{x\to-1^+}\frac{x+2}{x+1} = \text{"}\frac{1}{0^+}\text{"} = \infty \quad \left(\begin{matrix}x+2\to 1\\ x+1\to 0^+\end{matrix}\right)$$

Hence $x=-1$ is a VA.

bonus problem [5 points extra] Use the definition of the derivative to compute the derivative of the function $f(x) = \sin x$ at a number $x = a$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sin(a+h)-\sin(a)}{h}$$

Addition Formula for sine

$$= \lim_{h\to 0} \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(a)[\cos(h)-1]}{h} + \cos(a)\frac{\sin(h)}{h}$$

LL (sum)

$$= \lim_{h\to 0} \sin(a)\frac{[\cos(h)-1]}{h} + \lim_{h\to 0} \cos(a)\frac{\sin(h)}{h}$$

LL (const. mult.)

$$= \sin(a)\lim_{h\to 0} \frac{[\cos(h)-1]}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1} + \cos(a)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(a)\lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos(h)+1)} + \cos(a)\cdot 1$$

$\sin^2 h + \cos^2 h = 1$

$$= \sin(a)\lim_{h\to 0}\frac{1-\sin^2(h)-1}{h(\cos(h)+1)} + \cos(a)$$

LL (mult.)

$$= \sin(a)\left[\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\lim_{h\to 0}\frac{-\sin(h)}{\cos(h)+1}\right] + \cos(a)$$

DSP

$$= \sin(a)\left[1\cdot\frac{-0}{1+1}\right] + \cos(a)$$

$$= \cos(a)$$