9-9:50am **Midterm Exam 1**

Spring 2013 Math 0220

100 points total **S o l u t i o n s**

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1. [10 points] Calculate $y'(0)$ if $\sin(xy) = (x+1)^{3/2} - y$.

Solution: To find $y'(x)$ we differentiate the equation with respect to x

$$\frac{d}{dx}\left[\sin(xy) = (x+1)^{3/2} - y\right], \quad \cos(xy)(y + xy') = \tfrac{3}{2}(x+1)^{1/2} - y'.$$

When $x = 0$ we have $\cos(0) \cdot y(0) = \frac{3}{2} - y'(0)$, $y'(0) = \frac{3}{2} - y(0)$.

To find $y(0)$ we plug $x = 0$ into the original equation $\sin(0) = 1 - y(0)$, $y(0) = 1$.

Therefore, $y'(0) = \dfrac{3}{2} - 1 = \dfrac{1}{2}$.

2. Find the limit, probably infinite, if it exists. If the limit does not exist explain why. If it is infinity, then find its sign. Provide all the necessary steps or explanations. You can use any known method.

(a) [10 points] $L = \lim\limits_{x \to -2} \dfrac{\sqrt{x+3} - 1}{x+2}$

Solution: $$L = \lim_{x \to -2} \frac{\sqrt{x+3} - 1}{x+2} \cdot \frac{\sqrt{x+3} + 1}{\sqrt{x+3} + 1} = \lim_{x \to -2} \frac{x+3-1}{(x+2)\sqrt{x+3}+1}$$

$$= \lim_{x \to -2} \frac{x+2}{(x+2)\sqrt{x+3}+1} = \lim_{x \to -2} \frac{1}{\sqrt{x+3}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

(b) [10 points] $\lim\limits_{x \to 0} \dfrac{x^2}{|x|}$.

Solution: $$\lim_{x \to 0^-} \frac{x^2}{|x|} = \lim_{x \to 0^-} \frac{x^2}{-x} = \lim_{x \to 0^-} (-x) = 0$$

$$\lim_{x \to 0^+} \frac{x^2}{|x|} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^-} x = 0$$

Hence, $\lim\limits_{x \to 0} \dfrac{x^2}{|x|} = 0$.

(c) [10 points] $L = \lim\limits_{h \to 0} \dfrac{f(5+h) - f(5)}{h}$, where $f(x) = x^3$.

Solution: $L = f'(5),\ f'(x) = 3x^2$. Hence $L = 3 \cdot 25 = 75$.

3. [15 points] A screen saver displays the outline of a 3 cm by 2 cm rectangle and then expands the rectangle in such a way that the 2 cm side is expanding at the rate of 2 cm/sec and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are 12 cm by 8 cm?

Solution: Let's denote the smaller side by x and the larger one by y. Then $\frac{y}{x} = \frac{3}{2}$ or $y = \frac{3}{2}x$. The area is $A = xy = \frac{3}{2}x^2$. A and x both are functions of t (time). Then

$$\frac{dA}{dt} = \frac{3}{2} \cdot 2x\frac{dx}{dt} = 3x\frac{dx}{dt} = 3 \cdot 8 \cdot 2 = 48 \text{ cm}^2/\text{sec, since } \frac{dx}{dt} = 2 \text{ cm/sec.}$$

Another way: $A = xy$, Differentiating w.r.t. t and using the Product Rule: $\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$.

It is given that $y = \frac{3}{2}x$. Then $\frac{d}{dt}\left[y = \frac{3}{2}x\right]$ gives $\frac{dy}{dt} = \frac{3}{2} \cdot \frac{dx}{dt} = \frac{3}{2} \cdot 2 = 3$ cm/sec. Then

$$\frac{dA}{dt} = 2 \cdot 12 + 8 \cdot 3 = 24 + 24 = 48 \text{ cm}^2/\text{sec.}$$

4. [10 points] Find an equation of the normal line to the curve $y = \frac{\cos^2 x}{2}$ at the point $\left(\frac{\pi}{4}, \frac{1}{4}\right)$. Write the answer in the slope-intercept form.

Solution: The normal line equation is $y - \frac{1}{4} = -\frac{1}{m}\left(x - \frac{\pi}{4}\right)$, where $m = y'\left(\frac{\pi}{4}\right)$.

$$y'(x) = \frac{d}{dx}\left(\frac{\cos^2 x}{2}\right) = \frac{1}{2} \cdot 2\cos x(-\sin x) = -\cos x \sin x,\ y'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{1}{2}.$$

The normal line equation is $y - \frac{1}{4} = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x - \frac{\pi}{2} + \frac{1}{4}$.

5. Evaluate derivatives. Mention rules used.

(a) [10 points] Evaluate $f'(\pi)$ if $f(x) = \frac{x}{\cos x}$.

Solution: Quotient rule: $f'(x) = \frac{(x)'\cos x - x(\cos x)'}{\cos^2 x} = \frac{\cos x + x\sin x}{\cos^2 x}$

$$f'(\pi) = \frac{-1 + 0}{(-1)^2} = -1.$$

(b) [10 points] Evaluate $f''(3)$ if $f(x) = \sqrt{x^2 - 5}$.

Solution: $f(x) = (x^2 - 5)^{1/2}$. Chain rule: $f'(x) = \frac{1}{2}(x^2 - 5)^{-1/2} \cdot 2x = x(x^2 - 5)^{-1/2}$

Product and Chain rules: $f''(x) = 1 \cdot (x^2 - 5)^{-1/2} + x \cdot \left(-\frac{1}{2}\right)(x^2 - 5)^{-3/2} \cdot 2x$

$f''(x) = (x^2 - 5)^{-1/2} - x^2(x^2 - 5)^{-3/2}$

$f''(3) = 4^{-1/2} - 9 \cdot 4^{-3/2} = \dfrac{1}{2} - 9 \cdot \dfrac{1}{8} = \dfrac{4}{8} - \dfrac{9}{8} = -\dfrac{5}{8}$.

6. [15 points] Explain why the function

$$f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

is discontinuous at $x = 1$. Find a continuous function $g(x)$ such that $g(x) = f(x)$ when $x \neq 1$.

Solution: $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x^2 - x}{x^2 - 1} = \lim\limits_{x \to 1} \dfrac{x(x-1)}{(x-1)(x+1)} = \lim\limits_{x \to 1} \dfrac{x}{x+1} = \dfrac{1}{2}$.

$f(1) = 1$ and $\lim\limits_{x \to 1} f(x) \neq f(1)$. Hence, the function is discontinuous at $x = 1$.

$$g(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$$

or $g(x) = \dfrac{x}{x+1}$. (We should also assume that $x \neq -1$).

bonus problem. [10 points extra] Find $y''(2)$ if $x^3 + y^3 = 7$.

Solution: We differentiate the equation with respect to x to obtain

$\dfrac{d}{dx}\left[x^3 + y^3 = 8\right] \;\Rightarrow\; 3x^2 + 3y^2 y' = 0 \;\Rightarrow\; y' = -\dfrac{x^2}{y^2}$

If $x = 2$, then $y = -1$ and $y'(2) = -4$

Differentiate the equation one more time

$\dfrac{d}{dx}\left[3x^2 + 3y^2 y' = 0\right] \;\Rightarrow\; 6x + 6y(y')^2 + 3y^2 y'' = 0$

When $x = 2$, $y = -1$ we have

$6 \cdot 2 + 6(-1)(-4)^2 + 3(-1)^2 y''(2) = 0$, $6(2 - 16) + 3y''(2) = 0$ or $3y''(2) = 6 \cdot 14$,

$y''(2) = 28$.