9-9:50am **Midterm Exam 2**

Spring 2013 Math 0220

100 points total **S o l u t i o n s**

---

1. [15 points] Find $(f^{-1})'(3)$ if $f(x) = x^4 + 3x^3 - 1$. [Hint: $f(1) = 3$].

Solution: $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))}$. To use the formula we need to find $f'(x)$ and $f^{-1}(3)$.

$f^{-1}(3) = x \;\Leftrightarrow\; f(x) = 3 \;\Rightarrow\; x = 1$. $f'(x) = 4x^3 + 9x^2$.

Then $f'(f^{-1}(3)) = f'(1) = 4 + 9 = 13$ and $(f^{-1})'(3) = \dfrac{1}{13}$.

2. (a) [10 points] Find the linear approximation of the function $f(x) = x^{2/3}$ at $a = 64$.

Solution: The linearization at $a = 64$ is $L(x) = f(64) + f'(64)(x - a)$

$f(64) = 64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16,\quad f'(x) = \frac{2}{3}x^{-1/3},\quad f'(64) = \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}.$

Linear approximation at $a = 64$ is $f(x) \approx L(x) = 16 + \frac{1}{6}(x - 64) = \frac{1}{6}x + \frac{16}{3}$.

(b) [10 points] Use the linearization above to approximate the number $63^{2/3}$.

Solution: $63^{2/3} = f(63) \approx L(63) = 16 + \frac{1}{6}(63 - 64) = 16 - \frac{1}{6} = 15\frac{5}{6}$.

3. [15 points] A house was purchased for \$100,000 in 2002. Six years after its value was \$120,000. Find the value of the house in 2014 (exactly twelve years after the purchase) if it grows exponentially. The result is an integer number. Find this number. Do not leave fractions, logs or exponents in your answer.

Solution: Let t be the number of years passed after the purchase and $V(t)$ be the value of the house. We have $V(t) = V(0)e^{kt}$. It is given that $V(0) = 100,000$.

Then $V(6) = 100,000\left(e^k\right)^6 = 120,000,\; \left(e^k\right)^6 = \frac{12}{10} = \frac{6}{5},\; e^k = \left(\frac{6}{5}\right)^{1/6}$.

Hence, $V(t) = 100,000\left(\frac{6}{5}\right)^{t/6}$ and

$V(12) = 100,000\left(\frac{6}{5}\right)^{12/6} = 100,000\left(\frac{6}{5}\right)^2 = 100,000\cdot\frac{6}{5}\cdot\frac{6}{5} = 120,000\cdot\frac{6}{5} = \$144,000.$

4. Find the limits. Use l'Hospital rule if appropriate. In your solution mention types of all

indeterminate forms.

(a) [10 points] $\lim\limits_{x\to 0} \dfrac{x+\tan x}{\sin x}$.

Solution: $\lim\limits_{x\to 0} \dfrac{x+\tan x}{\sin x} \left[\text{"}\frac{0}{0}\text{"}\right] \overset{LR}{=} \lim\limits_{x\to 0} \dfrac{1+\sec^2 x}{\cos x} \overset{DSP}{=} \dfrac{1+1}{1} = 2.$

(b) [10 points] $\lim\limits_{x\to -\infty} x^2 e^x$.

Solution: $\lim\limits_{x\to -\infty} x^2 e^x \left[\text{"}\infty\cdot 0\text{"}\right] = \lim\limits_{x\to -\infty} \dfrac{x^2}{e^{-x}} \left[\text{"}\frac{\infty}{\infty}\text{"}\right] \overset{LR}{=} \lim\limits_{x\to -\infty} \dfrac{2x}{-e^{-x}} \left[\text{"}\frac{\infty}{\infty}\text{"}\right]$

$\overset{LR}{=} \lim\limits_{x\to -\infty} \dfrac{2}{e^{-x}} = \lim\limits_{x\to -\infty} 2e^x = 0.$

5. [15 points] Use optimization method to find a point on the line $y = 4 - x$ that is closest to the point $(3, 4)$

Solution: Let (x, y) be such a point on the line and d be the distance between (x, y) and $(3, 4)$ At the closest point d^2 attains its absolute minimum: $d^2 = (x-3)^2 + (y-4)^2 \to \min$. Hence the problem is to find x at which the function $f(x) = d^2 = (x-3)^2 + (4-x-4)^2 = (x-3)^2 + x^2 = 2x^2 - 6x + 9$ attains its absolute minimum value, where x can be any real number.

CNs: $f'(x)$ is defined everywhere. $f'(x) = 4x - 6 = 0$, $x = 1.5$ is the only CN.

$f''(x) = 4 > 0$ for all x, the graph of $f(x)$ is concave up everywhere and $f(x)$ attains its absolute minimum at the CN $x = 1.5$. Then $y = 4 - 1.5 = 2.5$.

Answer: the point is $(1.5, 2.5)$.

6. [15 points] For the equation $x^2 = 12$ use Newton's method with the initial approximation $x_1 = 3$ to find the third approximation x_3 to the positive root. (Write your answer as a reduced fraction).

Solution: $x^2 = 12 \Leftrightarrow x^2 - 12 = 0$. Let $f(x) = x^2 - 12$. To find an approximation of a root of the equation $x^2 = 12$ which is the root of the equation $f(x) = 0$ we apply Newton's method.

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, with $x_1 = 3$.

$f'(x) = 2x \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{x_n^2 - 12}{2x_n} = \dfrac{x_n^2 + 12}{2x_n} = \dfrac{x_n}{2} + \dfrac{6}{x_n}$

$x_2 = \dfrac{x_1}{2} + \dfrac{6}{x_1} = \dfrac{3}{2} + 2 = \dfrac{7}{2}$

$$x_3 = \frac{x_2}{2} + \frac{6}{x_2} = \frac{7}{4} + \frac{6 \cdot 2}{7} = \frac{49 + 48}{28} = \frac{97}{28}.$$

bonus problem. [15 points extra] Show that the equation $\sin x + 1 = 4x$ has exactly one real root.

Solution: The equation $\sin x - 1 = 4x$ is equivalent to the equation $\sin x - 4x + 1 = 0$. Consider the function $f(x) = \sin x - 4x + 1$. $f(0) = 1 > 0$, $f(\pi) = 0 - 4\pi + 1 < 0$. On the interval $[0, \pi]$ the function is continuous and satisties the inequality $f(0) > 0 > f(\pi)$. Applying the IVT with $N = 0$ we get that there is c in $(0, \pi)$ such that $f(c) = 0$, which means that c is a real root of the given equation.

Assume that the equation and hence the function has two or more roots. Let a and b be two roots. Then $f(a) = f(b) = 0$. The function is continuous and differentiable everywhere on $(-\infty, \infty)$. By the MVT there is c such that $f'(c) = 0$. But $f'(x) = \cos x - 4 \leq -3 < 0$ for all x. It contradicts to the fact that at c the derivative is zero. Hence the assumption about the existence of at least two roots was wrong. Therefore, the function and hence the equation has exactly one root.