Midterm Exam 2

Spring 2015 | **S o l u t i o n s** | Math 0220

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1. Use linear approximation to estimate the number $\frac{1}{2.01}$.

Solution: Consider the function $f(x) = \dfrac{1}{x} = x^{-1}$ and its linearization $L(x)$ at the point $x = 2$:

$$L(x) = f(2) + f'(2)(x-2)$$

We have $f(2) = \frac{1}{2}$, $f'(x) = -x^{-2}$, $f'(2) = -2^{-2} = -\frac{1}{4}$. Hence $L(x) = \frac{1}{2} - \frac{1}{4}(x-2)$

Then $\frac{1}{2.01} = f(2.01) \approx L(2.01) = \frac{1}{2} - \frac{1}{4}(2.01-2) = \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{100} = \frac{200}{400} - \frac{1}{400} = \frac{199}{400}$

2. The function $f(x) = 5 + x + 2\tan^{-1} x$, $-1 < x < 1$ is one-to-one. Find $\left(f^{-1}\right)'(5)$.

Solution: $\left(f^{-1}\right)'(5) = \dfrac{1}{f'\left(f^{-1}(5)\right)}$ $\qquad f(0) = 5$, hence $f^{-1}(5) = 0$

$f'(x) = 1 + \dfrac{2}{1+x^2}$, $\quad f'\left(f^{-1}(5)\right) = f'(0) = 1 + 2 = 3$, $\quad \left(f^{-1}\right)'(5) = \frac{1}{3}$

3. A bacteria culture initially ($t = 0$) contains 100 cells and grows at a rate proportional to its size. After three hours ($t = 3$) the population has increased to 500. How many cells there were two hours ($t = 2$) after the initial moment? Leave your answer in exact form.

Solution: Let $N(t)$ be the number of cells after t hours. Then $N(t) = 100a^t$.

We know that $N(3) = 500$. So, we have $100a^3 = 500$, $a^3 = 5$, $a = 5^{1/3}$.

Hence $N(t) = 100 \cdot 5^{t/3}$ and $N(2) = 100 \cdot 5^{2/3}$ cells.

4. Find the point on the line $y = -2x + 5$ that is closest to the origin. Use optimization method to solve the problem.

Solution: Let $(x, y) = (x, -2x+5)$ be a point on the line. The squared distance between the point and the origin $(0, 0)$ is given by the function $f(x) = (x-0)^2 + (-2x+5-0)^2 = 5x^2 - 20x + 25$. Absolute minimum of $f(x)$ gives absolute minimum of the distance.

CNs: $f'(x) = 10x - 20 = 0 \quad \Rightarrow \quad x = 2$; $\quad f'(x)$ is defined everywhere.

Hence the only CN is $x = 2$.

$f''(x) = 10 > 0$ and the function $f(x)$ is concave up on the entire number line (for all x). Therefore it has an absolute minimum at $x = 2$.

When $x = 2$ $y = -2 \cdot 2 + 5 = 1$. The closest point to the origin is $(2, 1)$

5. Find the limit, if it exists. If the limit does not exist explain why. You may use the L'Hospital's Rule.

(a) $\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{x^2}$

Solution: $$\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{x^2} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{x\to 0} \frac{2e^{2x} - 2}{2x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{x\to 0} \frac{4e^{2x}}{2} \overset{DSP}{=} 2$$

(b) $\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta - 1}{1 - \cos 4\theta}$

Solution: $$\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta - 1}{1 - \cos 4\theta} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{4\sin 4\theta} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{\theta\to\frac{\pi}{2}} \frac{-\sin\theta}{16\cos 4\theta} \overset{DSP}{=} \frac{-1}{16} = -\frac{1}{16}$$

(c) $\lim_{x\to 0^+} x^{\sqrt{x}}$

Solution: Denote $y = x^{\sqrt{x}}$. Then $\ln y = \sqrt{x}\ln x$ and

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{H}{=}} \lim_{x\to 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} -2\,x^{3/2}x^{-1} = -2 \lim_{x\to 0^+} x^{1/2} \overset{DSP}{=} 0$$

Therefore, $$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1$$

6. For the function $f(x) = 4x^3 - x^4$ make two sign diagrams: one for the first derivative that also contains information about f (CNs, increase/decrease, relative maximums and minimums), the other for the second derivative that also contains information about f (IPs, concavity).

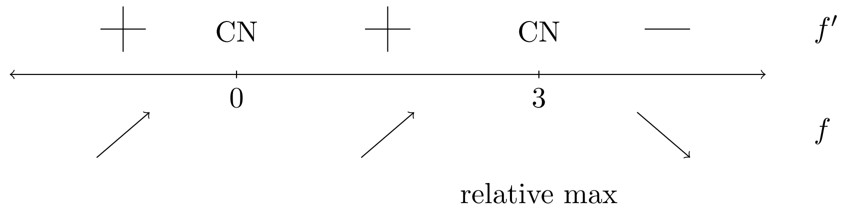
Solution: $f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$. f' is defined everywhere.

CNs are $x = 0$ and $x = 3$.

$f'(x) > 0$ on $(-\infty, 3)$ $f'(x) < 0$ on $(3, \infty)$

Correspondingly, $f(x)$ is increasing on $(-\infty, 3)$, $f(x)$ is decreasing on $(3, \infty)$

$f(x)$ has a relative maximum when $x = 3$ and no relative minimum.

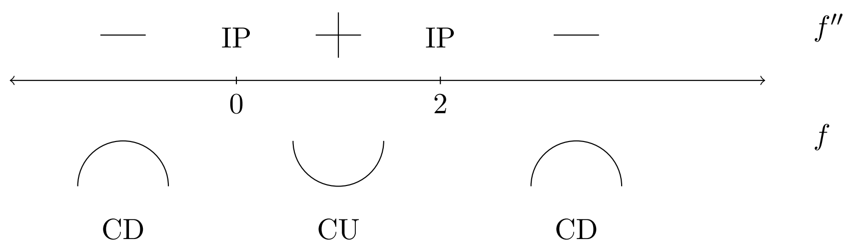


$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0.$

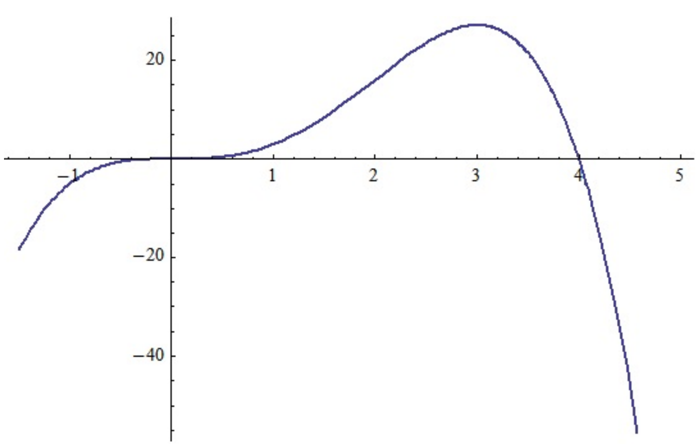
IPs are at $x = 0$ and $x = 2$. IPs: $(0, 0)$ and $(2, 16)$.

$f''(x) < 0$ on $(-\infty, 0)$ and $(2, \infty)$ $f''(x) > 0$ on $(0, 2)$

Correspondingly, $f(x)$ is concave down on $(-\infty, 0)$ and $(2, \infty)$, $f(x)$ is concave up on $(0, 2)$



This is the graph of the function



7. For the equation $x^2 = 6$ use Newton's method with the initial approximation $x_1 = 2$ to find the third approximation x_3 to the positive root. (Write your answer as a reduced fraction).

Solution: $x^2 = 6 \Leftrightarrow x^2 - 6 = 0$. Let $f(x) = x^2 - 6$. To find an approximation of a root of the equation $x^2 = 6$ which is the root of the equation $f(x) = 0$ we apply Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{with } x_1 = 2.$$

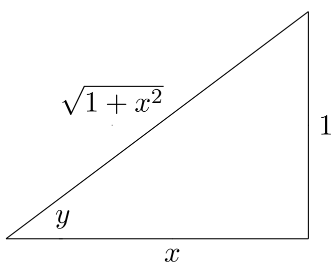
$$f'(x) = 2x \quad \Rightarrow \quad x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n} = \frac{x_n^2 + 6}{2x_n} = \frac{x_n}{2} + \frac{3}{x_n}$$

$$x_2 = \frac{x_1}{2} + \frac{3}{x_1} = 1 + \tfrac{3}{2} = \tfrac{5}{2}$$

$$x_3 = \frac{x_2}{2} + \frac{3}{x_2} = \tfrac{5}{4} + 3 \cdot \tfrac{2}{5} = \tfrac{5}{4} + \tfrac{6}{5} = \tfrac{25+24}{20} = \tfrac{49}{20}.$$

bonus problem Simplify the expression $\sin(\cot^{-1} x)$.

Solution: By the definition of an inverse function $y = \cot^{-1} x \quad \Leftrightarrow \quad x = \cot y$.



Or $\cot y = \dfrac{x}{1}$ (see the picture, y is an angle in the right triangle).

Then $\sin(\cot^{-1} x) = \sin y = \dfrac{1}{\sqrt{1+x^2}}$.