Math 0220 **Quiz 4**

Spring 2015 **S o l u t i o n s**

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1. (12 pm) Use a linear approximation to estimate the number $(8.03)^{2/3}$.

*Solution:* Consider the function $f(x) = x^{2/3}$ and its linearization $L(x)$ at the point $x = 8$:

$$L(x) = f(8) + f'(8)(x - 8)$$

We have $f(8) = 8^{2/3} = 4$, $f'(x) = \frac{2}{3}x^{-1/3}$, $f'(8) = \frac{2}{3} \cdot 8^{-1/3} = \frac{1}{3}$. Hence

$$L(x) = 4 + \frac{1}{3}(x - 8)$$

Then $(8.03)^{2/3} = f(8.03) \approx L(8.03) = 4 + \frac{1}{3}(8.03 - 8) = 4 + \frac{0.03}{3} = 4 + \frac{3}{300} = 4\frac{1}{100} = 4.01$

1. (1 pm) [5 points] Use differentials to estimate the amount (in $\mathrm{m}^3$) of paint needed to apply a coat of paint 0.03 cm thick to a ball with diameter 20 m.

*Solution:* $V(r) = \frac{4}{3}\pi r^3$, $V'(r) = \dfrac{dV}{dr} = 4\pi r^2$, $dV = 4\pi r^2\, dr$.

The amount of paint is $\Delta V \approx dV = 4\pi r^2\, dr \approx 4\pi r^2\, \Delta r$

$r = 10$ m, $\Delta r = 0.03 = 3 \cdot 10^{-2}$ cm $= 3 \cdot 10^{-4}$ m. Then

$\Delta V = 4\pi \cdot 10^2 \cdot 3 \cdot 10^{-4} = 12 \cdot 10^{-2} = 0.12\ \mathrm{m}^3$

2. (12 pm) [5 points] The half-life of cesium-137 is 30 years. If we have 200-mg sample then how much of it will remain after 50 years? Leave your answer in exact form.

*Solution:* Let $m(t)$ be the mass that remains after $t$ years. Then $m(t) = 200a^t$.

We know that $m(30) = 100$ and $m(30) = 200a^{30}$.

So, we have $200a^{30} = 100$, $a^{30} = \frac{1}{2}$, $a = \left(\frac{1}{2}\right)^{\frac{1}{30}}$.

Hence $m(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{30}}$ and $m(50) = 200\left(\frac{1}{2}\right)^{\frac{5}{3}} = 100\left(\frac{1}{2}\right)^{\frac{2}{3}}$ mg.

2. (1 pm) [5 points] Find $y'$ if $y = (\sin x)^x$.

*Solution:* $\ln y = x\ln(\sin x)$, $\dfrac{d}{dx}[\ln y] = \dfrac{d}{dx}[x\ln(\sin x)]$

$\dfrac{y'}{y} = \ln(\sin x) + x \cdot \dfrac{\cos x}{\sin x} = \ln(\sin x) + x\cot x$

$y' = y(\ln(\sin x) + x\cot x)$, $y' = (\sin x)^x(\ln(\sin x) + x\cot x)$

bonus problem (12 pm) [5 points extra] $f(x) = \sin^2 x + 5\cos^2 x$, $0 \le x \le \frac{\pi}{2}$. Find $\left(f^{-1}\right)'(4)$. Simplify your answer.

*Solution:* $\left(f^{-1}\right)'(4) = \dfrac{1}{f'\left(f^{-1}(4)\right)}$ $f(x) = \sin^2 x + \cos^2 x + 4\cos^2 x = 1 + 4\cos^2 x$

$1 + 4\cos^2 x = 4$, $4\cos^2 x = 3$, $\cos x = \frac{\sqrt{3}}{2}$, $x = \frac{\pi}{6} = f^{-1}(4)$

$f'(x) = -4\sin 2x$, $f'\left(f^{-1}(4)\right) = f'\left(\frac{\pi}{6}\right) = -2\sqrt{3}$, $\left(f^{-1}\right)'(4) = -\frac{1}{2\sqrt{3}}$