Math 0220 **Midterm Exam 1**

Fall 2017 **S o l u t i o n s**

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1. The graph of $y = f(x)$ is given.

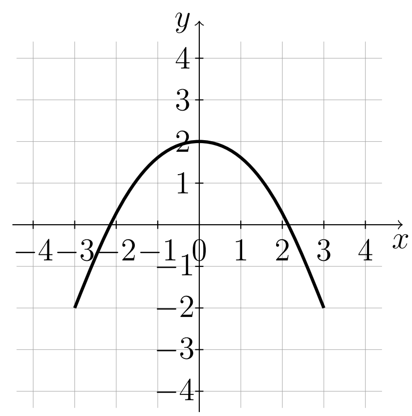


Figure 1: $y = f(x)$.

Draw $y = -\frac{1}{2}f(x+1)$ by applying three-step process:

1. draw $y = \frac{1}{2}f(x)$; 2. draw $y = -\frac{1}{2}f(x)$; 3. draw $y = -\frac{1}{2}f(x+1)$.

Solution:

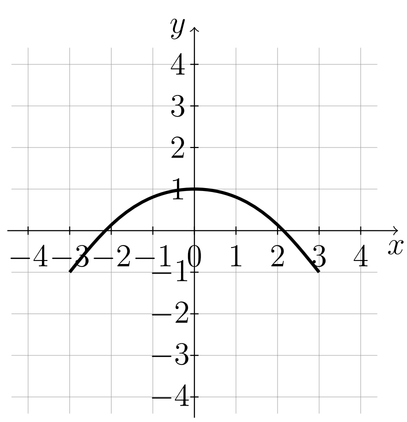


Figure 2: $y = \frac{1}{2}f(x)$

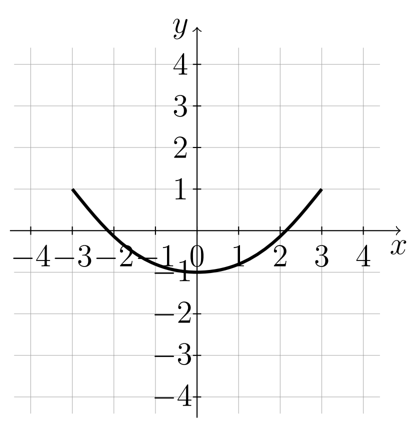


Figure 3: $y = -\frac{1}{2}f(x)$

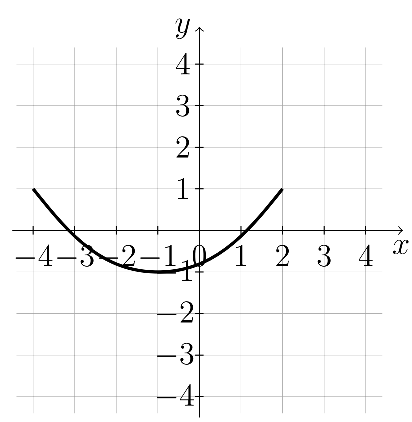


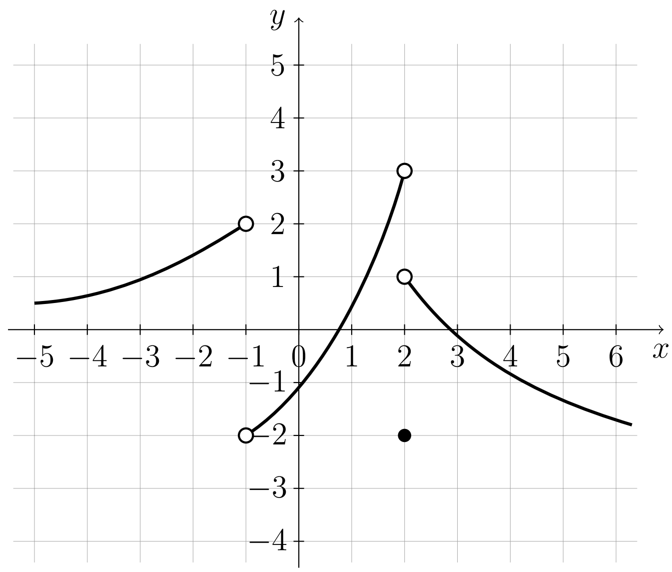
Figure 4: $y = -\frac{1}{2}f(x+1)$

2. Sketch the graph of an example of a function that satisfies all of the given conditions.

$\lim_{x\to -1^-} f(x) = 2,\quad \lim_{x\to -1^+} f(x) = -2,\quad f(-1)$ is undefined,

$\lim_{x\to 2^+} f(x) = 1,\quad \lim_{x\to 2^-} f(x) = 3,\quad f(2) = -2.$

Solution:



3. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \sin\left(\frac{2}{x}\right) = 0.$

Solution: $-1 \le \sin u \le 1$, for any real number u. Let $u = \frac{2}{x},\ x \ne 0.$

Then $-1 \le \sin\left(\frac{2}{x}\right) \le 1$ for all real x except $x = 0$.

We have $-x^2 \le x^2 \sin\left(\frac{2}{x}\right) \le x^2$ because $x^2 \ge 0$.

By the Squeeze Theorem $\lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2 \sin\left(\frac{2}{x}\right) \le \lim_{x\to 0} x^2$

$\lim_{x\to 0} -x^2 = 0,\quad \lim_{x\to 0} x^2 = 0.$

Therefore $0 \le \lim_{x\to 0} x^2 \sin\left(\frac{2}{x}\right) \le 0.$

So $\lim_{x\to 0} x^2 \sin\left(\frac{2}{x}\right) = 0.$

4. Find the limit $\lim_{x\to 3} \frac{2-x}{(x-3)^2}.$

Solution: Because x approaches 3 we can assume that x is the interval $I = [2.9, 3.1]$.

Inside I $\quad -1.1 \le 2-x \le -0.9 \quad \Rightarrow \quad 2-x<0.$

For any x inside the interval I $\quad (x-3)^2 \ge 0$ and hence $\dfrac{2-x}{(x-3)^2} < 0.$

As x is getting closer to 3 the expression $\dfrac{2-x}{(x-3)^2}$ can be made as large (in negative value) as we like.

Therefore $\displaystyle\lim_{x\to 0} \frac{2-x}{(x-3)^2} = -\infty.$

5. Find the derivatives of the function $f(x) = \sqrt{x-1}$ using the definition of derivative.

Solution: $\quad f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{x-1+h}-\sqrt{x-1}}{h} \cdot \frac{\sqrt{x-1+h}+\sqrt{x-1}}{\sqrt{x-1+h}+\sqrt{x-1}}$$

$$= \lim_{h\to 0} \frac{(x-1+h)-(x-1)}{h\left(\sqrt{x-1+h}+\sqrt{x-1}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x-1+h}+\sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x-1+h}+\sqrt{x-1}} = \frac{1}{\sqrt{x-1+0}+\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

6. Find the derivatives of following functions. Mention rules used. You do not need to simplify your answer.

(a) $f(x) = 2\pi$

Solution: $\quad f'(x) = 0$

Derivative of a constant is zero.

(b) $f(x) = \dfrac{x^3-3x-1}{\sqrt{x}}$

Solution: $\quad f(x) = \dfrac{x^3-3x-1}{x^{1/2}} = x^{5/2} - 3x^{1/2} - x^{-1/2}$

$f'(x) = \frac{5}{2}x^{3/2} - \frac{3}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$

Power Rule.

Alternative solution: $f'(x) = \dfrac{(3x^2-3)\sqrt{x} - \dfrac{x^3-3x-1}{2\sqrt{x}}}{x}$

Quotient and Power Rules.

(c) $g(t) = t^3 \sin t$

Solution: $g'(t) = 3t^2 \sin t + t^3 \cos t$

Product Rule, Power Rule.

(d) $h(x) = \dfrac{3x^2}{2+x^2}$

Solution: $h'(x) = \dfrac{6x(2+x^2) - (3x^2)(2x)}{(2+x^2)^2} = \dfrac{12x}{(2+x^2)^2}$

Quotient Rule, Power Rule.

(e) $f(t) = \sqrt[3]{1+\sec t}$

Solution: $f(t) = (1+\sec t)^{1/3}$

$f'(t) = \frac{1}{3}(1+\sec t)^{-2/3} \sec t \tan t$

Chain Rule, Power Rule, derivative of $\sec x$.

(f) $g(x) = \cos^2(5+x^3)$

Solution: $g'(x) = -2\cos(5+x^3)\sin(5+x^3)(3x^2)^{-2/3} = -3x^2 \cos(2(5+x^3))$

Chain Rule, Power Rule, derivative of $\cos x$.

7. Find an equation of the tangent line to the curve $2x^2 + xy + y^3 = 12$ at $(1,\, 2)$. Write the answer in the slope-intercept form.

Solution: The tangent line equation is $y - 2 = m\,(x-1)$, where $m = y'\,(1)$.

To find $y'(x)$ we use implicit differentiation $\dfrac{d}{dx}\left(2x^2 + xy + y^3\right) = \dfrac{d}{dx}\,(12)$.

$4x + y + xy' + 3y^2y' = 0$ We plug in $x = 1,\ y = 2$: $4 + 2 + y' + 12y' = 0.$

$13y' = -6 \qquad y' = -\frac{6}{13}.$

The tangent line equation is $y - 2 = -\frac{6}{13}\,(x-1)$ or $y = -\frac{6}{13}x + \frac{32}{13}.$

8. Find the linearization $L(x)$ of the function $f(x) = \cos\left(x + \frac{\pi}{2}\right)$ at $a = 0$ and use it to approximate the number $\cos\left(\frac{\pi}{2} - 0.01\right)$.

Solution: $L(x) = f(0) + f'(0)(x-0)$

We have $f(0) = \cos\left(\frac{\pi}{2}\right) = 0,\ f'(x) = -\sin\left(x + \frac{\pi}{2}\right),\ f'(0) = -\sin\left(\frac{\pi}{2}\right) = -1.$

Hence $L(x) = -x.$

Then $\cos\left(\frac{\pi}{2} - 0.01\right) = f(-0.01) \approx L(-0.01) = 0.01 \quad\Rightarrow\quad \cos\left(\frac{\pi}{2} - 0.01\right) \approx 0.01.$

bonus problem Find the limit $\lim\limits_{x\to 0} \dfrac{\cos(\pi + x) + 1}{\pi x}$ if it exists.

If the limit does not exist explain why. Show all work. No L'Hospital's Rule is allowed.

Solution: Denote $f(x) = \cos x.$

Then $\lim\limits_{x\to 0} \dfrac{\cos(\pi + x) + 1}{\pi x} = \frac{1}{\pi} \lim\limits_{h\to 0} \dfrac{\cos(\pi + h) - \cos\pi}{h} = \frac{1}{\pi} \lim\limits_{h\to 0} \dfrac{f(\pi + h) - f(\pi)}{h} = \frac{1}{\pi}\, f'(\pi).$

$f'(x) = -\sin x.$ Therefore $\lim\limits_{x\to 0} \dfrac{\cos(\pi + x) + 1}{\pi x} = -\frac{1}{\pi}\,\sin\pi = 0.$