Math 0290 **Midterm Exam 2**

Fall 2017 **S o l u t i o n s**

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1. The function $g(x) = 5 + 2x + e^x$ is one-to-one. Find $\left(g^{-1}\right)'(6)$.

Solution: $\left(g^{-1}\right)'(6) = \dfrac{1}{g'\left(g^{-1}(6)\right)}$ $\quad g(0) = 6$, hence $g^{-1}(6) = 0$.

$g'(x) = 2 + e^x$, $\quad g'\left(g^{-1}(6)\right) = g'(0) = 2 + 1 = 3$, $\quad \left(g^{-1}\right)'(6) = \frac{1}{3}$

2. Find the limit, if it exists. If the limit does not exist explain why. You may use the L'Hospital's Rule.

(a) $\displaystyle\lim_{x \to 0} \frac{e^{8x} - 1}{2x}$

Solution: $$\lim_{x \to 0} \frac{e^{8x} - 1}{2x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{LH}{=}} \lim_{x \to 0} \frac{8e^{8x}}{2} \overset{DSP}{=} \frac{8}{2} = 4$$

(b) $\displaystyle\lim_{x \to \infty} x^2 e^{-3x}$

Solution: $$\lim_{x \to \infty} x^2 e^{-3x} = \lim_{x \to \infty} \frac{x^2}{e^{3x}} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{LH}{=}} \lim_{x \to \infty} \frac{2x}{3e^{3x}} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{LH}{=}} \lim_{x \to \infty} \frac{2}{9e^{3x}}$$

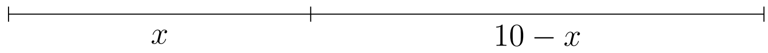
$$= \tfrac{2}{9} \lim_{x \to \infty} e^{-3x} = 0$$

(c) (10 points) $\displaystyle\lim_{x \to 0} \frac{x - \tan 2x}{\sin 3x}$

Solution: $$\lim_{x \to 0} \frac{x - \tan 2x}{\sin 3x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{LH}{=}} \lim_{x \to 0} \frac{1 - 2\sec^2 2x}{3\cos 3x} \overset{DSP}{=} \frac{1 - 2}{3} = -\tfrac{1}{3}$$

3. A piece of wire 10 meters long is to be cut into two pieces. Both pieces are bent into squares. How should the wire be cut so that the total area enclosed by the two squares is minimized? Use optimization method to solve the problem.

Solution: Let x be one piece of the wire. Then the other piece is $10 - x$.
$0 < x < 10$. We can assume that $0 \le x \le 10$.



Out of the piece with length x a square is made with the size $\frac{x}{4}$ and the area of $\left(\frac{x}{4}\right)^2$.
Out of the piece with length $10 - x$ a square is made with the size $\frac{10-x}{4}$ and the area of $\left(\frac{10-x}{4}\right)^2$.

Then the total area of two squares is $A(x) = \left(\dfrac{x}{4}\right)^2 + \left(\dfrac{10-x}{4}\right)^2 = \dfrac{x^2}{16} + \dfrac{(10-x)^2}{16}$.

CNs: $A'(x) = \dfrac{x}{8} - \dfrac{10-x}{8} = 0 \quad \Rightarrow \quad 2x - 10 = 0 \quad \Rightarrow \quad x = 5.$

$A(0) = \frac{25}{4}, \quad A(5) = \frac{50}{16} < \frac{25}{4} = \frac{100}{16}, \quad A(10) = \frac{25}{4}.$

[An alternative solution $A'(x)$ is defined everywhere. Hence the only CN is $x = 5$.
$A''(x) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} > 0$.]

Therefore A attains an absolute minimum at $x = 5$.

Answer: The wire should be cut in half.

4. Use Newton's method with the initial approximation $x_1 = 2$ to find the second approximation x_2 to the number $\sqrt{6}$. (Write your answer as a reduced fraction).

Solution: Denote $x = \sqrt{6}$. Keeping in mind that $x > 0$ this equation can be written as $x^2 = 6 \quad \Leftrightarrow \quad x^2 = 6 \quad \Leftrightarrow \quad x^2 - 6 = 0$. Let $f(x) = x^2 - 6$. To find an approximation of a root of the equation $x^2 = 6$ which is the root of the equation $f(x) = 0$ we apply Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{with } x_1 = 2.$$

$$f'(x) = 2x \quad \Rightarrow \quad x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n} = \frac{x_n^2 + 6}{2x_n} = \frac{x_n}{2} + \frac{3}{x_n}$$

$$x_2 = \frac{x_1}{2} + \frac{3}{x_1} = 1 + \tfrac{3}{2} = \tfrac{5}{2}.$$

5. For the function $f(x) = 2 + 3x - x^3$

(a) Find the intervals of increase or decrease.

Solution: $f'(x) = 3 - 3x^2 = 3(1-x)(1+x) = 0$. f' is defined everywhere.

CNs are $x = -1$ and $x = 1$.

$f'(x) > 0$ on $(-1, 1)$ $f'(x) < 0$ on $(-\infty, -1) \cup (1, \infty)$

Correspondingly, $f(x)$ is increasing on $(-1, 1)$,

$f(x)$ is decreasing on $(-\infty, -1) \cup (1, \infty)$

(b) Find the local maximum and local minimum values.

Solution: $f(x)$ has a local maximum when $x = 1$ and a local minimum when $x = -1$.

$f(-1) = 0$, $f(1) = 4$.

The local maximum value is 4 and the local minimum value is 0.

(c) Find the intervals of concavity and the inflection points.

Solution: $f''(x) = -6x$. f'' is defined everywhere.

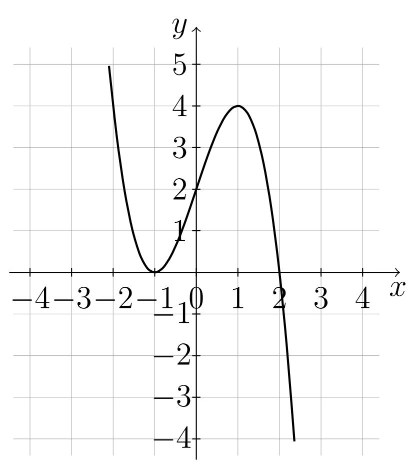
IP is at $x = 0$ (when $f''(x) = -6x = 0$). $f(0) = 2$. IP is $(0, 2)$.

$f''(x) > 0$ on $(-\infty, 0)$ $f''(x) < 0$ on $(0, \infty)$.

Correspondingly, $f(x)$ is concave up on $(-\infty, 0)$, $f(x)$ is concave down on $(0, \infty)$.

(d) Use the information from previous parts to sketch the graph of $f(x)$.

Solution:



6. Find antiderivative $F(x)$ for the function

(a) $f(x) = 2 + e^x.$

Solution: $F(x) = 2x + e^x + C.$

(b) $f(x) = \sqrt[3]{x} + \dfrac{1}{x}.$

Solution: $f(x) = x^{1/3} + \dfrac{1}{x}.$ $F(x) = \dfrac{1}{1+\frac{1}{3}}\,x^{1+\frac{1}{3}} + \ln|x| + C.$

$F(x) = \frac{3}{4}x^{4/3} + \ln|x| + C.$

(c) $f(x) = 3\sin x + \sec^2 x.$

Solution: $F(x) = -3\cos x + \tan x + C.$

bonus problem Evaluate the limit $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{2x}$

Solution: By the definition of the number e $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e.$

Then $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{2x} = \lim\limits_{x\to\infty} \left(\left(1+\frac{1}{x}\right)^{x}\right)^2 = \left(\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}\right)^2 = e^2.$