Math 0230 **Quiz 2**

Fall 2013 **S o l u t i o n s**

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1. [9 am class] Determine whether the integral $I = \int_{-6}^{2} \frac{dx}{\sqrt[3]{2-x}}$ is convergent or divergent. Then evaluate the integral if it is convergent and write your answer as a single number.

Solution: The integral is improper because $\frac{1}{\sqrt[3]{2-x}}$ is undefined at $x = 2$. Then

$$I = \lim_{t\to 2^-} \int_{-6}^{t} (2-x)^{-1/3}\,dx = \lim_{t\to 2^-} \left[-\frac{3}{2}(2-x)^{2/3}\right]_{-6}^{t} = \lim_{t\to 2^-} \left[-\frac{3}{2}(2-t)^{2/3} + \frac{3}{2}(8)^{2/3}\right]$$

$$= 0 + \frac{3}{2}\cdot 4 = 6$$

The integral is convergent and its value is 6.

1. [11 am class] Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ about the y-axis.

Solution: Draw a picture. The curves intersect at the points $(0,0)$ and $(1,1)$.

Hence, $0 \le x \le 1$, $r = x$, $\Delta r = \Delta x$, $h = \sqrt{x} - x^2$

$$\Delta V = 2\pi x\left(\sqrt{x} - x^2\right)\Delta x = 2\pi(x^{3/2} - x^3)\Delta x$$

$$V = 2\pi \int_0^1 (x^{3/2} - x^3)\,dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{2}{5} - \frac{1}{4}\right] = 2\pi\cdot\frac{3}{20} = \frac{3}{10}\pi = 0.3\,\pi$$

1. [1 pm class] Use the method of washers to find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ about the line $y = 2$.

Solution: The points of intersection are $x = 0$ and $x = 1$. Hence $0 \le x \le 1$. We use vertical rectangles. The cross-section is a washer with the outer radius $2 - x^2$ and the inner radius $2 - \sqrt{x}$. Then

$$\Delta V = \pi((2-x^2)^2 - (2-\sqrt{x})^2)\Delta x = \pi(4 - 4x^2 + x^4 - 4 + 4\sqrt{x} - x)\Delta x = \pi(x^4 - 4x^2 - x + 4x^{1/2})\Delta x$$

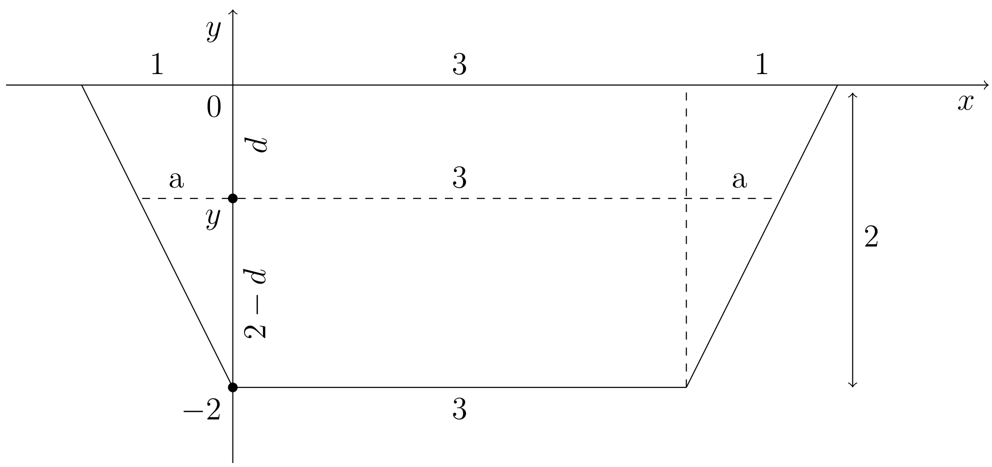
$$V = \pi\int_0^1 (x^4 - 4x^2 - x + 4x^{1/2})\,dx = \pi\left[\frac{x^5}{5} - \frac{4}{3}x^3 - \frac{x^2}{2} + \frac{8}{3}x^{3/2}\right]_0^1 = \pi\left[\frac{1}{5} - \frac{4}{3} - \frac{1}{2} + \frac{8}{3}\right] = \frac{31}{30}\pi$$

2. [9 am class] Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep. The weight density of water is $62.5 = 125/2$ lb/ft^3.

Solution: See Example 6 on the page 403.

2. [11 am class] A gate in an irrigation canal is in the form of a trapezoid 3 feet wide at the bottom, 5 feet wide at the top, with the height equal to 2 feet. It is placed vertically in the canal with the water extending to its top. Find the hydrostatic force in pounds on the gate. The weight density of water is $62.5 = 125/2$ lb/ft^3.

Solution: Let's place the coordinate axes as it is shown on the picture. We use horizontal strips of equal width δy and consider a strip on the depth $d = -y$ with $-2 \le y \le 0$ according to the picture.



By similar triangles: $\dfrac{a}{1} = \dfrac{2-d}{2} = \dfrac{2+y}{2}$, $a = \dfrac{2+y}{2}$.

The length of the strip is $l = 3 + 2a = 3 + 2 + y = 5 + y$ and its area is $\Delta A = (5+y)\,\Delta y$.

The pressure on the strip is $P = 62.5d = -62.5y$ (note that $d = -y$ since y is negative) and the force is

$\Delta F = P\Delta A = -62.5y(5+y)\,\Delta y = -62.5(5y + y^2)$.

Then the total force is

$$F = -\int_{-2}^{0} 62.5(5y + y^2)\,dy = -\frac{125}{2}\left[\frac{5}{2}y^2 + \frac{y^3}{3}\right]_{-2}^{0} = -\frac{125}{2}\left[0 - \frac{5}{2}\cdot 4 + \frac{8}{3}\right] = \frac{125\cdot 11}{3}$$

2. [1 pm class] Find the exact length L of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ when $1 \le x \le 2$.

Hint: $(x^2-1)^2 = x^4 - 2x^2 + 1$, $x^4 + 2x^2 + 1 = (x^2+1)^2$.

Show that the integrand is $\dfrac{1}{2}x + \dfrac{1}{2}x^{-1}$ and evaluate the corresponding integral.

Solution: $\dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x} = \dfrac{x^2-1}{2x}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x^2-1)^2}{(2x)^2} = 1 + \frac{x^4 - 2x^2 + 1}{4x^2} = \frac{x^4 + 4x^2 - 2x^2 + 1}{4x^2}$$

$$= \frac{x^4 + 2x^2 + 1}{4x^2} = \frac{(x^2+1)^2}{(2x)^2}.$$

$\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{x^2+1}{2x} = \dfrac{1}{2}x + \dfrac{1}{2}x^{-1}$. Hence,

$$L = \frac{1}{2}\int_1^2 \left(x + x^{-1}\right)\, dx = \frac{1}{2}\left[\frac{1}{2}x^2 + \ln x\right]_1^2 = \frac{1}{2}\left[2 + \ln 2 - \frac{1}{2} + 0\right] = \frac{3}{4} + \frac{\ln 2}{2}$$

bonus problem 1 [5 points extra] Find the exact length L of the curve $y = \sqrt{3 - x^2}$ when $-\sqrt{3} \le x \le 0$. You may use ANY method to find the right answer.

Solution: The curve is a quarter of a circle of radius $\sqrt{3}$. Therefore, $L = \dfrac{1}{4} \cdot 2\pi \left(\sqrt{3}\right)^2 = \dfrac{3\pi}{2}$.

bonus problem 2 [5 points extra] Evaluate the integral $I = \displaystyle\int_4^\infty \frac{x^{2/3}}{x^{3/2} - 2}\, dx$ if it is convergent.

Solution: Consider the integral $I_2 = \displaystyle\int_4^\infty x^{-5/6}\, dx$. I_2 is divergent because

$$I_2 = \lim_{t\to\infty} \int_4^t x^{-5/6}\, dx = \lim_{t\to\infty} 6x^{1/6} \Big|_4^t = \lim_{t\to\infty} 6\left(t^{1/6} - 4^{1/6}\right) = \infty$$

$$\frac{x^{2/3}}{x^{3/2} - 2} > \frac{x^{2/3}}{x^{3/2}} = x^{2/3 - 3/2} = x^{-5/6} > 0 \text{ when } x \ge 4.$$

By the comparison theorem I is divergent.