Math 0230 **Quiz 3**

Fall 2013 **S o l u t i o n s**

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1. [5 points] Find an equation of the tangent line to the parametric curve $x = t - t^{-1}$, $y = 1 + t^2$ at the point where $t = \dfrac{1}{2}$. Represent the equation in the slope-intercept form.

Solution: Let $t_0 = \dfrac{1}{2}$. Then $x_0 = x(t_0) = \dfrac{1}{2} - 2 = -\dfrac{3}{2}$, $y_0 = y(t_0) = 1 + \dfrac{1}{4} = \dfrac{5}{4}$,

$$\frac{dx}{dt}\bigg|_{t=t_0} = 1 + t^{-2}\Big|_{t=1/2} = 1 + 4 = 5, \qquad \frac{dy}{dt}\bigg|_{t=t_0} = 2t\Big|_{t=1/2} = 1.$$

The slope is $m = \dfrac{1}{5}$.

The tangent line equation is $y - \dfrac{5}{4} = \dfrac{1}{5}\left(x + \dfrac{3}{2}\right)$, $\quad y = \dfrac{1}{5}x + \dfrac{3}{10} + \dfrac{5}{4}$, $\quad y = \dfrac{1}{5}x + \dfrac{31}{20}$

2. [5 points] Find the area of the region that lies inside the curve $r = \sin\theta$ and outside the curve $r = 1 - \sin\theta$.

Solution: The first curve is a circle centered at the point $(0, 1/2)$ with radius $1/2$. The second curve is a cardioid.

Points of intersections: $\sin\theta = 1 - \sin\theta$, $\sin\theta = 1/2$, $\theta = \pi/6$ and $\theta = 5\pi/6$.

Hence,

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(\sin^2\theta - (1 - \sin\theta)^2\right) d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(\sin^2\theta - 1 + 2\sin\theta - \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (-1 + 2\sin\theta)\, d\theta = \frac{1}{2}\Big[-\theta\, - 2\cos\theta\Big]_{\pi/6}^{5\pi/6} = \frac{1}{2}\left(-\frac{5\pi}{6} + \sqrt{3} + \frac{\pi}{6} + \sqrt{3}\right) = \sqrt{3} - \frac{\pi}{3}$$

bonus problem [5 points extra] Find the exact length of the cardioid $r = 1 + \cos\theta$.

Solution: Let L be the length of the cardioid.

Consider the upper half of the cardioid when $0 \le \theta < \pi$. Denote its lenght by L_1.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + \cos\theta)^2 + (-\sin\theta)^2 = 2(1 + \cos\theta) = 4 \cdot \frac{1 + \cos\theta}{2} = 4\cos^2\frac{\theta}{2}$$

$\sqrt{4\cos^2\frac{\theta}{2}} = 2\cos\frac{\theta}{2}$ because $\cos\frac{\theta}{2} \geq 0$ when $0 \leq \frac{\theta}{2} < \frac{\pi}{2}$ that corresponds to $0 \leq \theta < \pi$.

Then

$$L_1 = \int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta = 4\sin\frac{\theta}{2}\Big|_0^{\pi} = 4$$

Due to symmetry $L = 2\,L_1 = 2 \cdot 4 = 8$.