Math 0230 **Quiz 5**

Fall 2013 **S o l u t i o n s**

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1. [5 points] Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

$$y' = \frac{y^2}{x^3}, \qquad y(1) = -1$$

Solution: It is a separable equation

$$\frac{dy}{dx} = \frac{y^2}{x^3}, \quad \int \frac{dy}{y^2} = \int \frac{dx}{x^3}, \quad -\frac{1}{y} = -\frac{2}{x^2} + C, \quad \frac{1}{y} = \frac{2}{x^2} - C = \frac{2 - Cx^2}{x^2}.$$

$$y(x) = \frac{x^2}{2 - Cx^2}.$$

The initial condition: $y(1) = \dfrac{1}{2 - C} = -1, \quad C = 3.$

The solution is $\boxed{y(x) = \dfrac{x^2}{2 - 3x^2}}$.

2. [5 points] Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

$$\frac{y'}{x} - 2y = 2e^{x^2}, \qquad y(1) = -2$$

Solution: It is a linear differential equation $y' - 2xy = 2xe^{x^2}$.

The integrating factor is $I(x) = e^{\int -2x\,dx} = e^{-x^2}$. Then

$$e^{-x^2}y' - 2xe^{-x^2}y = 2x, \quad \left(e^{-x^2}y\right)' = 2x, \quad e^{-x^2}y = \int 2x\,dx = x^2 + C.$$

The general solution is $y(x) = (x^2 + C)\,e^{x^2}$.

The initial condition: $y(0) = C = -2$.

The solution is $\boxed{y(x) = (x^2 - 2)\,e^{x^2}}$.

bonus problem [5 points extra] Solve the initial-value problem with the second-order nonhomogeneous differential equation using the method of undetermined coefficients.

$$y'' + y = \cos x, \qquad y(0) = 0, \quad y'(\pi) = \frac{\pi}{2}$$

Solution: The characteristic equation of the complementary equation is

$r^2 + 1 = 0$. Its discriminant $D = 0 - 4 \cdot 1 = -4$ is negative, $\alpha = 0,\ \beta = \dfrac{\sqrt{4}}{2} = 1.$

The solution of the complementary equation is $y_c(x) = c_1 \cos x + c_2 \sin x$.

For a particular solution we cannot use $y_p(x) = A\cos x + B\sin x$ because it is a solution of the complementary equation.

So, we try $y_p(x) = Ax\cos x + Bx\sin x$. Then

$y_p{}' = A\cos x - Ax\sin x + B\sin x + Bx\cos x,\quad y_p'' = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x.$

Substitution in the differential equation gives

$y_p'' + y_p = 2A\sin x + 2B\cos x = \cos x,\quad A = 0,\ B = \dfrac{1}{2}.$

The general solution $y(x) = y_c(x) + y_p(x)$ is $y(x) = c_1\cos x + c_2\sin x + \dfrac{x\sin x}{2}$

The initial conditions give $y(0) = c_1 = 0,$

$y'(x) = c_2\cos x + \dfrac{\sin x + x\cos x}{2},\quad y'(\pi) = -c_2 + \dfrac{-\pi}{2} = \dfrac{\pi}{2},\quad c_2 = -\pi.$

The solution is $\boxed{y(x) = -\pi\sin x + \dfrac{x\sin x}{2}}$.