Math 0230 **Midterm Exam 1**

Fall 2014 **S o l u t i o n s**

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1. (15 points) Use Simpson's Rule to approximate the integral $I = \int_0^2 \frac{2}{1+x}\,dx$ if $n = 4$.

   Simplify your answer.

   Solution: $\Delta x = \frac{2-0}{4} = \frac{1}{2},\ x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2,$

   $f(x_0) = 2,\ f(x_1) = \frac{4}{3},\ f(x_2) = 1,\ f(x_3) = \frac{4}{5},\ f(x_4) = \frac{2}{3}.$

   $$I \approx S_4 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

   $$= \frac{1}{6}\left[2 + 4\cdot\frac{4}{3} + 2\cdot 1 + 4\cdot\frac{4}{5} + \frac{2}{3}\right] = \frac{1}{6}\left[4 + \frac{18}{3} + \frac{16}{5}\right] = \frac{1}{6}\left[4 + 6 + \frac{16}{5}\right]$$

   $$= \frac{1}{6}\left[10 + \frac{16}{5}\right] = \frac{1}{6}\cdot\frac{66}{5} = \frac{11}{5}$$

2. (15 points) Evaluate the improper integral $I = \int_4^5 \frac{6x}{\sqrt{x^2-16}}\,dx$

   if it is convergent or show that it is divergent.

   Solution: The integral is improper since the integrand has discontinuity at $x = 4$.

   $$I = \lim_{t\to 4^+} \int_t^5 \frac{6x}{\sqrt{x^2-16}}\,dx$$

   Subs: $u = x^2 - 16,\ du = 2x\,dx,\ 6x\,dx = 3\,du,\ u(t) = t^2 - 16,\ u(5) = 9.$

   $$I = \lim_{t\to 4^+} 3 \int_{t^2-16}^{9} u^{-1/2}\,du = 6 \lim_{t\to 4^+} u^{1/2}\Big|_{t^2-16}^{9} = 6 \lim_{t\to 4^+}\left(3 - (t^2-16)^{1/2}\right) = 6(3-0) = 18$$

   The value is finite, so the integral is convergent.

3. (15 points) Use the method of washers to find the volume of the solid obtained by rotating the region bounded by the curves $x = 2\sqrt{y}$ and $x = y$ about the line y-axis.

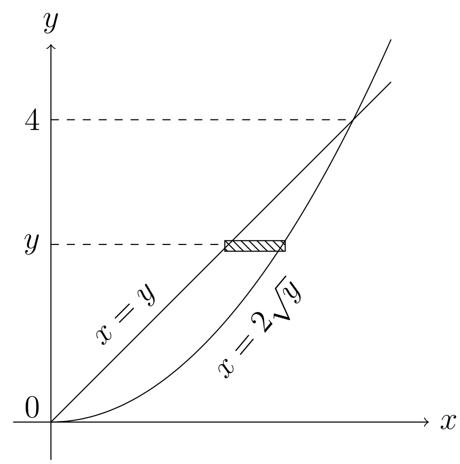
Solution:

To find points of intersection we solve the equation $2\sqrt{y} = y \quad \Leftrightarrow \quad 4y = y^2,\ y \geq 0 \quad \Leftrightarrow \quad y(y-4) = 0,$ $y \geq 0 \quad \Leftrightarrow \quad y = 0,\ y = 4$. Hence $0 \leq y \leq 4$.

We use horizontal rectangles. The cross-section is a washer with the outer radius $2\sqrt{y}$ and the inner radius y. Then

$$V = \pi \int_0^4 ((2\sqrt{y})^2 - y^2)\, dx = \pi \int_0^4 (4y - y^2)\, dx$$

$$= \pi \left[2y^2 - \frac{y^3}{3}\right]_0^4 = \pi \left[32 - \frac{64}{3}\right] = \frac{32}{3}\pi$$



4. (15 points) Find the exact length L of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ when $1 \leq x \leq 2$.

Hint: $1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \left(\frac{x}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x}\right)^2 = \left(\frac{x}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$

Solution: $\dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2, \qquad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{x}{2} + \frac{1}{2x}.$$

$$L = \frac{1}{2}\int_1^2 \left(x + \frac{1}{x}\right) dx = \frac{1}{2}\left[\frac{1}{2}x^2 + \ln x\right]_1^2 = \frac{1}{2}\left[2 + \ln 2 - \frac{1}{2} + 0\right] = \frac{3}{4} + \frac{\ln 2}{2}$$

5. (15 points) Find a unit vector that is orthogonal to both vectors $\bar{\mathsf{i}} - \bar{\mathsf{j}} + \bar{\mathsf{k}}$ and $\bar{\mathsf{i}} - \bar{\mathsf{k}}$.

Solution: Let $\bar{a} = \bar{\mathsf{i}} - \bar{\mathsf{j}} + \bar{\mathsf{k}} = \langle 1, -1, 1 \rangle$ and $\bar{b} = \bar{\mathsf{i}} - \bar{\mathsf{k}} = \langle 1, 0, -1 \rangle$. The vector

$$\bar{v} = \bar{a} \times \bar{b} = \begin{vmatrix} \bar{\mathsf{i}} & \bar{\mathsf{j}} & \bar{\mathsf{k}} \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 2, 1 \rangle$$

is orthogonal to both $\bar{a}$ and $\bar{b}$. The corresponding unit vector is

$$\bar{u} = \frac{\bar{v}}{|\bar{v}|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle \qquad \left(\text{or} \quad \left\langle -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle \right)$$

6. (10 points) Are vectors $\bar{a} = \langle 0, 2, 3 \rangle$, $\bar{b} = \langle 4, 1, -2 \rangle$, and $\bar{c} = \langle 8, 4, -1 \rangle$ coplanar?

Solution: To answer the question we find the triple product of the vectors:

$$\bar{a} \cdot \left(\bar{b} \times \bar{c} \right) = \begin{vmatrix} 0 & 2 & 3 \\ 4 & 1 & -2 \\ 8 & 4 & -1 \end{vmatrix} = 0 \cdot (-1 + 8) - 2 \cdot (-4 + 16) + 3 \cdot (16 - 8)$$

$$= 0 - 24 + 24 = 0$$

The triple product is zero. Therefore, the vectors are coplanar.

7. (10 points) Show that the lines

$$L_1 : \quad \frac{x}{3} = y = \frac{z - 1}{2} \qquad \text{and} \qquad L_2 : \quad x = s,\ y = s + 2,\ z = -s$$

are skew lines.

Solution: The parametric form of L_1 is $x = 3t,\ y = t,\ z = 2t + 1$.

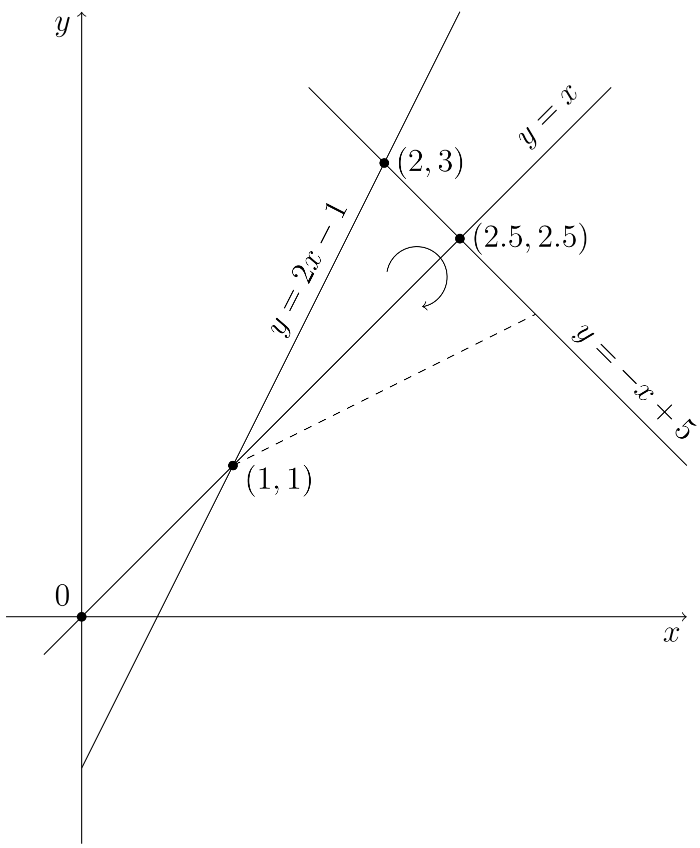
Direction vectors of both lines are $\bar{v}_1 = \langle 3, 1, 2 \rangle$ and $\bar{v}_2 = \langle 1, 1, -1 \rangle$.
$\bar{v}_2$ is not a constant multiple of $\bar{v}_1$. Hence the vectors and the lines are not parallel.
Now we assume that the lines intersect. Then in the point of intersection the following equalities must hold

$$3t = s, \qquad t = s + 2, \qquad 2t + 1 = -s$$

From the first two equalities we obtain $t = 3t + 2$ that gives $t = -1$, $s = -3$. When we plug these results into the third equality we get $-1 = 3$ which is not true. This condtradiction shows that our assumption was wrong and the lines do not intersect. Therefore, L_1 and L_2 are neither parallel nor they intersect. So, they are skew lines.

bonus problem (10 points extra) A region in the shape of a triangle is bounded by the lines $y = 2x - 1$, $y = x$, and $y = -x + 5$. Find the volume of the solid obtained by rotating the region about the line $y = x$. Use any method.

Solution: First, we find points of intersections (vertices of the triangle): the lines $y = 2x - 1$ and $y = x$ intersect at the point $(1, 1)$, the lines $y = x$ and $y = -x + 5$ intersect at the point $(2.5, 2.5)$, the lines $y = 2x - 1$ and $y = -x + 5$ intersect at the point $(2, 3)$.



The lines $y = x$ and $y = -x + 5$ are orthogonal because their slopes are negative reciprocals. Therefore, the solid is a cone with the volume $V = \frac{1}{3}\pi r^2 h$, where r is the distance between points $(2, 3)$ and $(2.5, 2.5)$, h is the distance between points $(1, 1)$ and $(2.5, 2.5)$.

Hence, $r^2 = (0.5)^2 + (0.5)^2 = 0.5 = \frac{1}{2}$, $h = \sqrt{(1.5)^2 + (1.5)^2} = 1.5\sqrt{2} = \frac{3}{2}\sqrt{2}$ and

$$V = \frac{1}{3}\pi \cdot \frac{1}{2} \cdot \frac{3}{2}\sqrt{2} = \frac{\pi\sqrt{2}}{4}.$$