Math 0230 **Midterm Exam 2**

Fall 2014 **S o l u t i o n s**

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1. (15 points) Find an equation of the tangent line to the parametric curve $x = \ln t + 1$, $y = t^2 + 2$ at the point $(1, 3)$ without eliminating the parameter. Write the equation in the slope-intercept form.

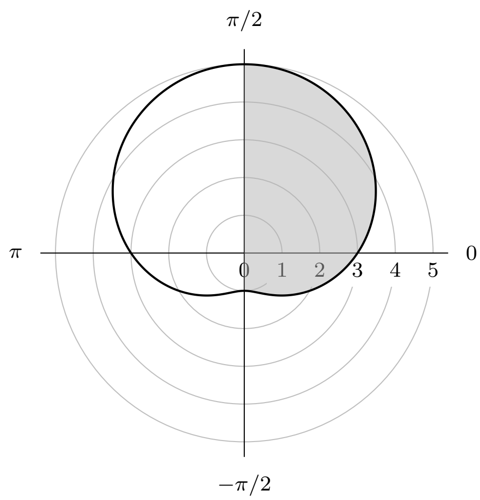
Solution: At $(1, 3)$ $x = \ln t + 1 = 1$, $\ln t = 0$, $t = 1$ and $y = t^2 + 2 = 1 + 2 = 3$. Hence $t = 1$.

$$\left.\frac{dx}{dt}\right|_{t=1} = \left.\frac{1}{t}\right|_{t=1} = \frac{1}{1} = 1, \qquad \left.\frac{dy}{dt}\right|_{t=1} = \left.2t\right|_{t=1} = 2 \cdot 1 = 2.$$

The slope is $m = \dfrac{2}{1} = 2$.

The tangent line equation is $y - 3 = 2\,(x - 1)$, $y = 2x - 2 + 3$, $y = 2x + 1$

2. (15 points) Find the area of the shaded region.



$r = 3 + 2\sin\theta$

Solution: $$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\,(3 + 2\sin\theta)^2\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (9 + 12\sin\theta + 4\sin^2\theta)\,d\theta$$

$$= \int_{0}^{\pi/2} (9 + 4\sin^2\theta)\,d\theta + 6\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta \qquad [9 + 4\sin^2\theta \text{ is even function}]$$

$$= \int_0^{\pi/2} (9 + 2(1 - \cos 2\theta))\, d\theta + 0 \qquad [\sin\theta \text{ is odd function}]$$

$$= \int_0^{\pi/2} (11 - 2\cos 2\theta)\, d\theta = \Big[11\theta \; - \sin 2\theta\Big]_0^{\pi/2} = \frac{11\pi}{2}$$

3. Determine whether the series is convergent or divergent. If it is convergent, find its limit.

(a) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}}$

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} = \sum_{n=1}^{\infty} 2\frac{2^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$

It is a geometric series with $a = 2$ and $r = \frac{2}{3}$, $|r| < 1$.

Hence, the series is convergent and its limit is $\dfrac{2}{1 - \frac{2}{3}} = 6$.

(b) (10 points) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{3}$

Solution: $a_n = \sqrt[n]{3}$, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{3} = \lim_{n\to\infty} 3^{\frac{1}{n}} = 3^{\lim_{n\to\infty} \frac{1}{n}} = 3^0 = 1 \neq 0$.

Hence, the series is divergent by the test of divergence.

4. (10 points) Find the limit of the sequence $\quad a_n = \dfrac{3 + 5n^2}{n + n^2}$

Solution: $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 + 5n^2}{n + n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{3}{n^2} + 5}{\frac{1}{n} + 1} = \frac{\lim_{n\to\infty} \frac{3}{n^2} + \lim_{n\to\infty} 5}{\lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} 1} = \frac{0 + 5}{0 + 1} = 5$

5. (15 points) Use the Integral Test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

Solution: Consider $f(x) = \dfrac{1}{\sqrt[3]{x}} = x^{-1/3}$, $x \geq 1$ which is a continuous, positive, and decreasing function. The last is true because $f'(x) = -\frac{1}{3}x^{-4/3} = -\dfrac{1}{3x^{4/3}} < 0$ when $x \geq 1$.

$$\int_1^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_1^{t} x^{-1/3}\, dx = \lim_{t\to\infty} \left[\frac{3}{2}\, x^{2/3}\right]_1^t = \frac{3}{2} \lim_{t\to\infty} (t^{2/3} - 1) = \infty.$$

The integral is divergent. Therefore, the series is also divergent by the Integral Test.

6. (10 points) Find a power series representation for the function and determine its radius of convergence.

$$f(x) = \frac{2}{3-x}$$

Solution: $f(x) = \frac{2}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n x^n = \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} x^n = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n.$

The series converges when $\left|\frac{x}{3}\right| < 1$, that is, when $|x| < 3$. Therefore, $R = 3$.

Alternative method to find R: $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{2}{3^{n+1}} \cdot \frac{3^{n+2}}{2}\right| = \lim_{n\to\infty} 3 = 3.$

7. (15 points) Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \frac{2}{x}$ centered at the number $a = 1$.

Solution: $f(x) = 2x^{-1}$, $f(1) = 2$, $f'(x) = -2x^{-2}$, $f'(1) = -2$, $f''(x) = 4x^{-3}$, $f''(1) = 4$, $f'''(x) = -12x^{-4}$, $f'''(1) = -12$.

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!} (x-1)^n = 2 - \frac{2}{1!}(x-1) + \frac{4}{2!}(x-1)^2 - \frac{12}{3!}(x-1)^3$$

$$T_3(x) = 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$$

bonus problem (15 points extra) In the Maclaurin series expansion for the function

$$f(x) = 42 \sin x \cos x$$

find the coefficient c_7 for the term containing x^7. Simplify your result.

Solution: $f(x) = 42 \sin x \cos x = 21 \sin 2x = 21 \left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right]$

The desired coefficient is

$$c_7 = -21 \cdot \frac{2^7}{7!} = -\frac{21 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = -\frac{2 \cdot 2 \cdot 2}{3 \cdot 5} = -\frac{8}{15}$$