Math 0230 Calculus 2 Lectures

Chapter 6 Techniques of Integration

Numeration of sections corresponds to the text

James Stewart, Essential Calculus, Early Transcendentals, Second edition.

Section 5.5 The Substitution Rule

Assume we have a composite function $f(u(x))$. Then the Substitution Rule for an integration is

$$\int f(u(x)) \cdot u'(x)\,dx = \int f(u)\,du$$

since $du = d[u(x)] = \dfrac{d}{dx}[u(x)]\,dx = u'(x)dx$

A similar rule for a definite integral is

$$\int_a^b f(u(x)) \cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

The Substitution Rule is also caled an u-substitution.

To recognize when to apply the Substitution Rule look first at an integrand. If it is a product of two parts one of which is a formula containing $u(x)$ and the other is the derivative of $u(x)$ up to a constant multiple then there is a big chance that the Substitution Rule will work.

Example 1. Evaluate the integral $I = \displaystyle\int 2\sin 2x\,dx$ by using the Substitution Rule.

Solution: $I = \displaystyle\int \sin 2x \cdot 2\,dx$. It is clear, that $(2x)' = 2$ or $d(2x) = 2dx$. The integrand is a product of two parts one of which is a formula (the fuction sin) containing $u(x) = 2x$ and the other is the derivative of $u(x)$ which is 2. Hence we can apply the Substitution Rule: $u = 2x, \quad du = 2dx$. Then

$I = \displaystyle\int \sin u\,du = -\cos u + C = -\cos 2x + C$. At the end we used so called back substitution. In other words,

$$I = \int 2\sin 2x\,dx = \int \sin 2x \cdot 2\,dx = \int \sin(2x)\,d(2x) = \int \sin u\,du = -\cos u + C = -\cos 2x + C.$$

Example 2. Evaluate the integral $I = \displaystyle\int \sin 5x\,dx$ by using the Substitution Rule.

Solution: Here we act similarly to what we did in the pervious example except there is no an extra constant multiple 5. To get it we multiple the integral by 1 in the form 5/5:

$$I = \frac{5}{5}\int \sin 5x\,dx = \frac{1}{5}\int \sin 5x \cdot 5\,dx = \frac{1}{5}\int \sin(5x)\,d(5x) = \frac{1}{5}\int \sin u\,du \;\; [u = 5x]$$

$$= -\frac{1}{5}\cos u + C = -\frac{\cos 5x}{5} + C.$$

Another way: u-substitution $u = 5x$, $du = 5dx$, $dx = du/5$. Then

$$I = \int \sin 5x\,dx = \int \sin u \cdot \frac{du}{5} = \frac{1}{5}\int \sin u\,du = -\frac{1}{5}\cos u + C = -\frac{\cos 5x}{5} + C$$

Example 3. Evaluate the integral $I = \int \frac{(\ln x)^3}{x}\,dx$.

Solution: We recognize the integrand as a product of two parts: one is $(\ln x)^3$ which is a formula containing $\ln x$ and the other $\frac{1}{x}$ is the derivative of $\ln x$. Hence we can apply the Substitution Rule.

Way 1: $u = \ln x$, $du = (\ln x)'dx = \frac{dx}{x}$, $dx = xdu$. Then

$$I = \int \frac{u^3}{x}\,xdu = \int u^3\,du = \frac{u^4}{4} + C = \frac{(\ln x)^4}{4} + C.$$

Way 2: $I = \int (\ln x)^3 \cdot \frac{1}{x}\,dx = \int (\ln x)^3 \cdot d(\ln x) = \int u^3\,du \;\; [u = \ln x] \;\; = \frac{u^4}{4} + C = \frac{(\ln x)^4}{4} + C.$

Example 4. Evaluate the definite integral $I = \int\limits_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution: We recognize the integrand as a product of two parts: one is $e^{\sqrt{x}}$ which is a formula containing $\sqrt{x}$ and the other $\frac{1}{\sqrt{x}}$ is the derivative of $\sqrt{x}$ up to a constant multiple 2. Hence we can apply the Substitution Rule:

$u = \sqrt{x}$, $du = (\sqrt{x})'dx = \frac{1}{2\sqrt{x}}\,dx = \frac{dx}{2\sqrt{x}}$, $dx = 2\sqrt{x}\,du = 2u\,du$, $u(1) = \sqrt{1} = 1$, $u(9) = \sqrt{9} = 3$. Then

$$I = \int\limits_1^3 \frac{e^u}{u} \cdot 2u\,du = 2\int\limits_1^3 e^u\,du = 2e^u\Big|_1^3 = 2(e^3 - e).$$

Symmetry

If f is continuous and even on $[-a, a]$, then $\int\limits_{-a}^{a} f(x)\,dx = 2\int\limits_0^a f(x)\,dx$.

If f is continuous and odd on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 5. Evaluate the definite integral $I = \int_{-\pi/4}^{\pi/4} \tan^5 x\,dx$.

Solution: The function $f(x) = \tan^5 x$ is continuous and odd on $[-\pi/4, \pi/4]$. Indeed, $f(-x) = (\tan(-x))^5 = (-\tan x)^5 = -\tan^5 x = -f(x)$. Hence $I = 0$.

Section 6.1 Integration by Parts

$(uv)' = u'v + uv'$, $d(uv) = du \cdot v + udv = vdu + udv$

$\int d(uv) = \int vdu + \int udv$, $uv = \int vdu + \int udv$

Integration by parts formula: $\int udv = uv - \int vdu$

LIATE rule: L = log, I = inverse trig, A = algebraic, T = trig, E = exponent

Example 1. Evaluate the definite integral $I = \displaystyle\int x \cos 3x \, dx$.

Solution: Integration by parts:

According to LIATE rule $u = x$ and $dv = \cos 3x \, dx$. Hence $du = dx$, $v = \dfrac{\sin 3x}{3}$. Then

$$I = \frac{x \sin 3x}{3} - \int \frac{\sin 3x}{3} \, dx = \frac{x \sin 3x}{3} + \frac{\cos 3x}{9} + C$$

Example 2. Evaluate the definite integral $I = \displaystyle\int_{-1}^{2} \ln(3x+4) \, dx$.

Solution: Note, the Substitution Rule is not applicable since the antiderivative of $\ln x$ is not elementary. We try integration by parts:

According to LIATE rule $u = \ln(3x+4)$ and $dv = dx$. Hence $du = \dfrac{3}{3x+4} \, dx$, $v = x$. Then

$$I = \int_{-1}^{2} u \, dv = \ln(3x+4) \cdot x \Big|_{-1}^{2} - \int_{1}^{3} x \cdot \frac{3}{3x+4} \, dx = 2 \ln 10 - \int_{-1}^{2} \frac{3x+4-4}{3x+4} \, dx$$

$$= 2 \ln 10 - \int_{-1}^{2} dx + 4 \int_{-1}^{2} \frac{1}{3x+2} \, dx$$

To evaluate the last integraal we use the substitution $u = 3x + 4$.

$du = 3dx$, $dx = \dfrac{du}{3}$, $u(-1) = 1$, $u(2) = 10$. Then

$$\int_{-1}^{2} \frac{1}{3x+4} \, dx = \int_{1}^{10} \frac{1}{u} \frac{du}{3} = \frac{1}{3} \ln |u| \Big|_{1}^{10} = \frac{1}{3} \ln 10$$

Hence, $I = 2 \ln 10 - \displaystyle\int_{-1}^{2} dx + \frac{4}{3} \ln 10 = 3\frac{1}{3} \ln 10 - 3$

Section 6.2 Trigonometric Integrals and Substitution

Integrals containing $\sin x$ and $\cos x$

Trig identity $\cos^2 x + \sin^2 x = 1$. Derivatives: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

1. Odd power: (a) $\cos^{2n+1} x\, dx = \cos^{2n} x \cos x\, dx = \left(\cos^2 x\right)^n \, d(\sin x)$

$= \left(1 - \sin^2 x\right)^n \, d(\sin x) = \left(1 - u^2\right)^n \, du, \quad u = \sin x$

(b) $\sin^{2n+1} x\, dx = \sin^{2n} x \sin x\, dx = \left(\sin^2 x\right)^n \, (d(-\cos x)) = \left(1 - \cos^2 x\right)^n \, (-d(\cos x))$

$= -\left(1 - u^2\right)^n \, du, \quad u = \cos x$

Example 1. Evaluate the integral $I = \displaystyle\int \sin^5 x\, dx$.

Solution: $\sin^5 x\, dx = \sin^4 x \sin x\, dx = \left(\sin^2 x\right)^2 \, (d(-\cos x)) = -\left(1 - \cos^2 x\right)^2 \, d(\cos x)$

$= -\left(1 - u^2\right)^2 \, du = \left(-1 + 2u^2 - u^4\right) du, \quad u = \cos x$. Then

$$I = \int \left(-1 + 2u^2 - u^4\right) du = -u + 2/3u^3 - u^5/5 + C = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C.$$

2. Odd power for one of trig functions (sin or cos) times the other rised to any power:

(a) $\cos^{2n+1} x \sin^m x\, dx = \cos^{2n} x \sin^m x \cos x\, dx = \left(1 - \sin^2 x\right)^n \sin^m x\, d(\sin x)$

$= \left(1 - u^2\right)^n u^m\, du, \quad u = \sin x$

(b) $\sin^{2n+1} x \cos^m x\, dx = -\left(1 - u^2\right)^n u^m\, du, \quad u = \cos x$

3. Even power. Use half-angle trig identity: $\cos^2 x = \dfrac{1 + \cos 2x}{2}$, $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

(a) $\cos^{2n} x\, dx = \left(\cos^2 x\right)^n x\, dx = \left(\dfrac{1 + \cos 2x}{2}\right)^n \, dx$

(b) $\sin^{2n} x\, dx = \left(\sin^2 x\right)^n x\, dx = \left(\dfrac{1 - \cos 2x}{2}\right)^n \, dx$

Example 2. Evaluate the integral $I = \displaystyle\int_0^{\pi/2} \cos^2 x\, dx$.

Solution: $I = \dfrac{1}{2} \displaystyle\int_0^{\pi/2} (1 + \cos 2x) \, dx = \dfrac{1}{2} \left[x + \dfrac{1}{2} \sin 2x\right]_0^{\pi/2} = \dfrac{\pi}{4}$

Integrals containing $\tan x$ and $\sec x$

Trig identity $\sec^2 x = 1 + \tan^2 x$.

Derivatives: $(\tan x)' = \sec^2 x$, $(\sec x)' = \sec x \tan x$,

$(\ln|\sec x|)' = \tan x$, $(\ln|\sec x + \tan x|)' = \sec x$.

4. $\tan x$ is in any power, $\sec x$ in even power:

$\tan^m x \sec^{2n+2} x\, dx = \tan^m x \sec^{2n} x \sec^2 x\, dx = \tan^m x (1+\tan^2 x)^n\, d(\tan x)$

$= u^m (1+u^2)^n\, du$, $u = \tan x$

Example 3. Evaluate the integral $I = \displaystyle\int \tan^3 x \sec^4 x\, dx$.

Solution: $\tan^3 x \sec^6 x\, dx = \tan^3 x \sec^4 x \sec^2 x\, dx = \tan^3 x (1+\tan^2 x)^2\, d(\tan x) = u^3(1+u^2)^2\, du$,

$u = \tan x$.

$$I = \int \left(u^3 + 2u^5 + u^7\right)\, du = \frac{u^4}{4} + \frac{u^6}{3} + \frac{u^8}{8} + C = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{3} + \frac{\tan^8 x}{8} + C.$$

5. $\tan x$ is an odd power, $\sec x$ in any power:

$\tan^{2n+1} x \sec^m x\, dx = \tan^{2n} x \sec^{m-1} x \sec x \tan x\, dx = (\sec^2 x - 1)^n \sec^{m-1} x\, d(\sec x)$

$= (u^2-1)^n u^{m-1}\, du$, $u = \sec x$

Example 4. Evaluate the integral $I = \displaystyle\int \tan^3 x\, dx$.

Solution: $\tan^3 x\, dx = \tan x \tan^2 x\, dx = \tan x(\sec^2 x - 1)\, dx = \tan x \sec^2 x\, dx - \tan x\, dx$.

$$I = \int \tan x \sec^2 x\, dx - \int \tan x\, dx = \int \sec x \tan x \sec x\, dx - \ln|\sec x|$$

$$= \textstyle\int \sec x\, d(\sec x) - \ln|\sec x| = \dfrac{\sec^2 x}{2} - \ln|\sec x| + C.$$

Trigonometric substitution

Expression	Substitution	Identity	Reason
$\sqrt{1-x^2}$	$x = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$	Domain: $1-x^2 \ge 0$ or $-1 \le x \le 1$ in accordance with $-1 \le \sin\theta \le 1$
$\sqrt{1+x^2}$	$x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$	Domain: $-\infty < x < \infty$ in accordance with $-\infty < \tan\theta < \infty$
$\sqrt{x^2-1}$	$x = \sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$	Domain: $x^2-1 \ge 0$, $x \le -1$ or $x \ge 1$ in accordance with $\sec x \le -1$ or $\sec x \ge 1$

In general,

Expression	Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

Example 5. Evaluate the integral $I = \displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

Solution: Trig substitution: $x = 2\sin\theta$, $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$, $dx = 2\cos\theta\,d\theta$. Then

$$I = \int \frac{4\sin^2\theta}{2\cos\theta} 2\cos\theta\,d\theta = 4\int \sin^2\theta\,d\theta = 4\int \frac{1-\cos 2\theta}{2}\,d\theta = 2\theta - \sin 2\theta + C$$

$$= 2\sin^{-1}(x/2) - 2\sin\theta\cos\theta + C = 2\sin^{-1}(x/2) - x\cos\theta + C$$

We need to express $\cos\theta$ through x. For that we use a right triangle with one of the angles is θ. Let the hypotenuse be 2. Then the side opposite to θ is $2\sin\theta = x$ and the side adjacent to θ is $\sqrt{4-x^2} = 2\cos\theta$. Hence $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$. Then

$$I = 2\sin^{-1}(x/2) - \frac{x\sqrt{4-x^2}}{2} + C$$

Section 6.3 Partial Fractions

Let $Q(x)$ be a polynomial. It is called reducible if it can be factored as a product of polynomials of smaller degrees. Otherwise it is called irredusible. For example, $Q_1(x) = x^3 - x^2 - x - 2 = (x-2)(x^2+x+1)$ and hence $Q_1(x)$ is reducible. On the other hand $Q_2(x) = x^2+x+1$ is irredusible since it is a polynomial of degree 2 which descriminant $D = 1-4 = -3$ is negative and it does not have roots. If a polynomial $Q(x)$ of degree k has k roots x_1, x_2, ... x_k then it can be factored $Q(x) = a(x-x_1)(x-x_2)\cdots(x-x_k)$, where a is its leading coefficient. The fundamental theorem of algebra says, that any polynomial can be factored into a product of irreducible polynomials of degree not higher than 2. Some of the factors can repeat, e.g. $x^3 - 3x^2 + 4 = (x-2)(x-2)(x+1) = (x-2)^2(x+1)$. In general, for any polinomial we have

$$Q(x) = (x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_k)^{n_k}(a_1x^2+b_1x+c_1)^{m_1}\cdots(a_lx^2+b_lx+c_l)^{m_l}$$

for some positive integers n, k, m, and l.

Let's denote the degree of a polynomial $Q(x)$ by $\deg Q$. Any rational function can be represented as $f(x) = \dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$, where $\deg R < \deg Q(!)$ and $\deg S = \deg P - \deg Q$ (there is no $S(x)$ if $\deg P \leq \deg Q$).

To simplify a process of integration of a rational function we represent $\dfrac{R(x)}{Q(x)}$ as a sum of partial fractions:

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_k)^{n_k}(a_1x^2+b_1x+c_1)^{m_1}\cdots(a_lx^2+b_lx+c_l)^{m_l}} =$$

$$\frac{R_1(x)}{(x-x_1)^{n_1}} + \frac{R_2(x)}{(x-x_2)^{n_2}} + \cdots + \frac{R_k(x)}{(x-x_k)^{n_k}} + \frac{R_{k+1}(x)}{(a_1x^2+b_1x+c_1)^{m_1}} + \cdots + \frac{R_{k+l}(x)}{(a_lx^2+b_lx+c_l)^{m_l}}$$

$$\frac{R_1(x)}{(x-x_1)^{n_1}} = \frac{A_1}{x-x_1} + \frac{A_2}{(x-x_1)^2} + \cdots + \frac{A_{n_1}}{(x-x_1)^{n_1}}$$

$$\frac{R_{k+1}(x)}{(a_1x^2+b_1x+c_1)^{m_1}} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2} + \cdots + \frac{A_{m_1}x+B_{m_1}}{(a_1x^2+b_1x+c_1)^{m_1}}$$

Note: the degree of top polynomial by one less then the degree of the bottom polynomial. This means that $\dfrac{R(x)}{Q(x)}$ can be represented as a sum of terms of two types.

Example 1. Express $\dfrac{x^2-8x-2}{x^3-x^2-x-2}$ as a sum of partial fractions.

Solution: $\dfrac{x^2-8x-2}{x^3-x^2-x-2} = \dfrac{x^2-8x-2}{(x-2)(x^2+x+1)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+x+1}$

$$= \frac{Ax^2+Ax+A+Bx^2-2Bx+Cx-2C}{(x-2)(x^2+x+1)}$$

Polynomials on tops must be equal:

$Ax^2 + Ax + A + Bx^2 - 2Bx + Cx - 2C = x^2 - 8x - 2$. It is true only if

$A + B = 1$, $A - 2B + C = -8$, $A - 2C = -2$. Solutions are $A = -2$, $B = 3$, $C = 0$. Hence,

$$\frac{x^2 - 8x - 2}{x^3 - x^2 - x - 2} = \frac{-2}{x - 2} + \frac{3x}{x^2 + x + 1}$$

Example 2. Evaluate the integral $I = \displaystyle\int \frac{x^2 - 8x - 2}{x^3 - x^2 - x - 2}\,dx.$

Solution: Using partial fractions from the previous example we obtain two simplier integrals

$$I = \int \left(\frac{-2}{x - 2} + \frac{3x}{x^2 + x + 1} \right) dx = -2 \int \frac{dx}{x - 2} + 3 \int \frac{x}{x^2 + x + 1}\,dx$$

To evaluate an integral $\displaystyle\int \frac{dx}{x - 2}\,dx$ use the substitution $u = x - 2$. Then $dx = du$ and

$$\int \frac{dx}{x - 2}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x - 2| + C$$

To evaluate an integral $\displaystyle\int \frac{x}{x^2 + x + 1}\,dx$ complete the square on the bottom

$$x^2 + x + 1 = \left(x + \frac{1}{2} \right)^2 + \frac{3}{4}$$

and then make the substitution $u = x + \dfrac{1}{2}$, $x = u - \dfrac{1}{2}$, $du = dx$. Then

$$\int \frac{x}{x^2 + x + 1}\,dx = \int \frac{x}{(x + 1/2)^2 + 3/4}\,dx = \int \frac{u - 1/2}{u^2 + (\sqrt{3}/2)^2}\,du$$

$$= \int \frac{u\,du}{u^2 + (\sqrt{3}/2)^2} - \frac{1}{2} \int \frac{du}{u^2 + (\sqrt{3}/2)^2}$$

For the first integral we use the substitution $v = u^2 + (\sqrt{3}/2)^2$, $dv = 2u\,du$, $u\,du = \dfrac{dv}{2}$. Then

$$\int \frac{u\,du}{u^2 + (\sqrt{3}/2)^2} = \frac{1}{2} \int \frac{dv}{v} = \frac{1}{2} \ln|v| + C = \frac{1}{2} \ln|u^2 + (\sqrt{3}/2)^2| + C$$

$$= \frac{1}{2} \ln|(x + 1/2)^2 + 3/4| + C = \frac{1}{2} \ln|x^2 + x + 1| + C$$

The second integral is expressed in terms of $\tan^{-1}$

Finally, $I = -2\ln|x - 2| + \dfrac{3}{2} \ln|x^2 + x + 1| - \dfrac{3}{2} \cdot \dfrac{2}{\sqrt{3}} \tan^{-1}\left(\dfrac{2u}{\sqrt{3}} \right) + C$

$$= -2\ln|x - 2| + \frac{3}{2} \ln|x^2 + x + 1| - \sqrt{3}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}} \right) + C$$

Section 6.5 Approximate Integration

Sometimes while evaluating a definite integral it is impossible to find antiderivative. In this situation we use approximate integration which arises from Riemann Sum. Different ways in setting the Riemann sum gives us different methods or rules.

Assume we want to evaluate the definite integral $\int_a^b f(x)\,dx$. We divide the interval $[a, b]$ into n subintervals each of the same lenght $\Delta x = \frac{b-a}{n}$. Let x_i^* be any point in the ith subinterval $[x_{i-1}, x_i]$. Then we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If x_i^* is the left end-point of each interval, then we get left endpoint approximation L_n. If x_i^* is the right end-point of each interval, then we get right endpoint approximation R_n. If x_i^* is the midpoint $\bar{x}_i$ of each interval, then we get

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$.

An approximation that is an average of leftpoint and rightpoint rules is called

Trapezoid Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $x_i = a + i\Delta x$.

If we use a parabola rather then a line segment for a top of each rectangle over an interval $[x_{i-1}, x_i]$, then we obtain

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n has to be even.

In practice, $S_n \succ T_n \succ M_n \succ L_n$ and R_n, in general, where the symbol $\succ$ means "is better than".

Example 1. Use the Midpoint Rule, the Trapezoid Rule, and Simpson's Rule with $n = 6$ to approximate the integral $I = \int_2^5 \sqrt[3]{x^2 - 2}\, dx$.

Solution: $\Delta x = \dfrac{5-2}{6} = 0.5$. Then

$x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, $x_4 = 4$, $x_5 = 4.5$, $x_6 = 5$ and

$\bar{x}_1 = 2.25$, $\bar{x}_2 = 2.75$, $\bar{x}_3 = 3.25$, $\bar{x}_4 = 3.75$, $\bar{x}_5 = 4.25$, $\bar{x}_6 = 4.75$

$M_6 = 0.5(f(2.25) + f(2.75) + f(3.25) + f(3.75) + f(4.25) + f(4.75)) = 0.5(1.4522 + 1.7718 + 2.0458 + 2.2934 + 2.5231 + 2.7396) = 6.4130$

$T_6 = \dfrac{0.5}{2}(f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)) = 0.25(1.2599 + 3.2396 + 3.8259 + 4.3445 + 4.8203 + 5.2656 + 2.8439) = 6.3999$

$S_6 = \dfrac{0.5}{3}(f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)) = 0.25(1.2599 + 6.4792 + 3.8259 + 8.6890 + 4.8203 + 10.5313 + 2.8439) = 6.4082$

More exact value gives $S_{30} = 6.4087$.

Example 2. Use the Midpoint Rule, the Trapezoid Rule, and Simpson's Rule with $n = 4$ to approximate the integral $I = \int_0^{\pi} x^2 \cos x\, dx$.

Section 6.6 Improper Integrals

Example 1. Evaluate the integral $I = \int_1^{\infty} \frac{1}{x^2}\, dx$.

Solution: Cosider areas under the graph on the intervals $[1,t]$, where $b = 2$, 10, 100, etc:

$$A_t = \int_1^t x^{-2}\, dx = -\frac{1}{x}\Big|_1^b = -(1/t - 1) = \frac{t-1}{t}$$

$A_2 = \frac{2-1}{2} = \frac{1}{2}$, $A_{10} = 0.9$, $A_{100} = 0.99$, and so on.

Improper Integral of Type I

If $\int_a^t f(x)\, dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$$

provided this limit exists (as a finite number)

If $\int_t^b f(x)\, dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x)\, dx$$

provided this limit exists (as a finite number)

The improper integrals $\int_a^{\infty} f(x)\, dx$ and $\int_{-\infty}^{b} f(x)\, dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

If both $\int_a^{\infty} f(x)\, dx$ and $\int_{-\infty}^{a} f(x)\, dx$ are convergent, then

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_a^{\infty} f(x)\, dx,$$

where a can be chosen arbitrary.

Example 2. Evaluate the integral $I = \int_1^\infty \frac{1}{(2x+1)^2}\,dx.$

Solution: $I = \lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^2}\,dx =$ [subs: $u = 2x+1$, $du = 2dx$, $u(1) = 3$, $u(t) = 2t+1$]

$$\lim_{t\to\infty} \frac{1}{2} \int_3^{2t+1} u^{-2}\,du = -\frac{1}{2} \lim_{t\to\infty} \frac{1}{u}\Big|_3^{2t+1} = -\frac{1}{2} \lim_{t\to\infty} \left(\frac{1}{2t+1} - \frac{1}{3}\right) = -\frac{1}{2}\left(-\frac{1}{3}\right) = \frac{1}{6}$$

Example 3. Evaluate the integral $I = \int_{-\infty}^0 e^{5x}\,dx.$

Solution: $I = \lim_{t\to-\infty} \int_t^0 e^{5x}\,dx = \lim_{t\to-\infty} \frac{1}{5} e^{5x} \Big|_t^0 = \frac{1}{5} \lim_{t\to-\infty} \left(e^0 - e^{5t}\right) = \frac{1}{5}\cdot 1 = \frac{1}{5}$

Discontinuous Integrands

Example 4. Evaluate the integral $I = \int_0^1 \frac{1}{\sqrt{x}}\,dx.$

Solution: The function $f(x) = \frac{1}{\sqrt{x}}$ is undefined at $x = 0$. Draw the graph.

Cosider areas under the graph on the intervals $[t, 1]$, where $t = 1/4,\ 1/100,\ 1/10000$, etc:

$$A_t = \int_t^1 x^{-1/2}\,dx = 2\sqrt{x}\ \Big|_t^1 = 2(1 - \sqrt{t})$$

$A_{1/4} = 2(1 - 1/2) = 1$, $A_{1/100} = 2(1 - 1/10) = 1.8$, $A_{1/10000} = 2(1 - 1/100) = 1.98$, and so on.

Improper Integral of Type II

If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number)

If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number)

The improper integrals $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

If $f(x)$ is continuous on $[a, b]$ and is discontinuous at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

Example 5. Evaluate the integral $I = \int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution: Discontinuity is at 0. Hence,

$$I = \lim_{t \to 0^+} \int_t^1 x^{-1/2}\,dx = \lim_{t \to 0^+} 2\sqrt{x}\,\Big|_t^1 = \lim_{t \to 0^+} 2(1 - \sqrt{t}) = 2$$

Example 6. Evaluate the integral $I = \int_{-1}^1 \frac{e^x}{e^x - 1}\,dx$.

Solution: Discontinuity is at 0. Hence,

$I = \lim_{t \to 0^-} \int_{-1}^t \frac{e^x}{e^x - 1}\,dx + \lim_{t \to 0^+} \int_t^1 \frac{e^x}{e^x - 1}\,dx = \lim_{t \to 0^-} \ln|e^x - 1|\,\Big|_{-1}^t + \lim_{t \to 0^+} \ln|e^x - 1|\,\Big|_t^1 = \lim_{t \to 0^-} (\ln|e^t - 1| - \ln(1 - e^{-1})) + \lim_{t \to 0^+} (\ln(e - 1) - \ln|e^t - 1|)$

Limits $\lim_{t \to 0^-} \ln|e^t - 1|$ and $\lim_{t \to 0^+} \ln|e^t - 1|$ are divergent. Hence I is divergent.

Comparison Theorem

Suppose that f and g are continuous functions on $[a, \infty)$ with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int\limits_a^\infty f(x)\,dx$ is convergent, then $\int\limits_a^\infty g(x)\,dx$ is convergent.

If $\int\limits_a^\infty g(x)\,dx$ is divergent, then $\int\limits_a^\infty f(x)\,dx$ is divergent.