Math 0230 Calculus 2 Lectures

Chapter 8 Series

Numeration of sections corresponds to the text

James Stewart, Essential Calculus, Early Transcendentals, Second edition.

Section 8.1 Sequences

A sequence is a list of numbers written in a definite order. It also can be treated as a function defined at the integer numbers only. A sequence can be finite or infinite.

Notation: $a_1,\ a_2,\ a_3,\ \ldots\ ,\ a_n,\ \ldots \quad \{a_1, a_2, a_3, \ldots\} \quad \{a_n\} \quad \{a_n\}_{n=1}^{\infty}$

$\{a_4, a_5, a_6, \ldots\} \quad \{a_n\}_{n=4}^{\infty}$

Example 1. (a) $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty} \qquad a_n = \dfrac{1}{n}, \qquad \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, \dfrac{1}{n}, \ldots\right\}$

(b) $\left\{\sin\dfrac{\pi n}{3}\right\}_{n=0}^{\infty} \qquad a_n = \sin\dfrac{\pi n}{3}, \qquad \left\{0, \dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}, 0, \ldots, \sin\dfrac{\pi n}{3}, \ldots\right\}$

(c) $\left\{\dfrac{(-1)^n n^2}{2^n}\right\}_{n=2}^{\infty} \qquad a_n = \dfrac{(-1)^n n^2}{2^n}, \qquad \left\{1, -\dfrac{9}{8}, 1, -\dfrac{25}{32}, \dfrac{9}{16}, \ldots, \dfrac{(-1)^n n^2}{2^n}, \ldots\right\}$

(d) The Fibonacci sequence $\{f_n\}$ cannot be represented by a formula and is defined recursively:

$f_1 = 1,\ \ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$ for $n \geq 3, \quad \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

Example 2. Find a formula for a general term of the sequence $\left\{1, \dfrac{4}{9}, \dfrac{5}{27}, \dfrac{2}{27}, \dfrac{7}{243}, \ldots\right\}$.

Solution: $\quad a_n = \dfrac{n+2}{3^n}$

As we can see the terms of this sequence are getting smaller and smaller closser to 0.

Definition A sequence $\{a_n\}$ has the limit L:

$$\lim_{n\to\infty} a_n = L$$

if the terms a_n can be made as close to L as we like by taking n sufficiently large. If the limit exists, then the sequence is called convergent, otherwise it's called divergent.

If $\lim\limits_{n\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim\limits_{n\to\infty} a_n = L$

Definition $\lim_{n\to\infty} a_n = \infty$ means that for any positive number M there is a positive integer N such that if $n > N$, then $a_n > M$.

Limit Laws
If a_n and b_n are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if} \ \ p > 0 \quad \text{and} \quad a_n > 0$$

Squeeze Theorem If $a_n \le b_n \le c_n$ for $n \ge N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

such that if $n > N$, then $a_n > M$.

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Example 3. Find $\lim_{n\to\infty} \dfrac{n+1}{3n-2}$.

Solution: Divide numerator and denominator by the highest power of n that occurs in the denominator and then apply the Limit Laws

$$\lim_{n\to\infty} \frac{n+1}{3n-2} = \lim_{n\to\infty} \frac{1+1/n}{3-2/n} = \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 1/n}{\lim_{n\to\infty} 3 - 2\lim_{n\to\infty} 1/n} = \frac{1+0}{3-0} = \frac{1}{3}$$

Example 4. Find $\lim_{n\to\infty} \dfrac{(-1)^n}{n+1}$.

Solution: $\lim_{n\to\infty} \left|\dfrac{(-1)^n}{n+1}\right| = \lim_{n\to\infty} \dfrac{1}{n+1}$

$0 < \dfrac{1}{n+1} < \dfrac{1}{n}$ when $n \ge 1$. $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

Then by Squeeze theorem $\lim_{n\to\infty} \dfrac{1}{n+1} = 0$ and $\lim_{n\to\infty} \dfrac{(-1)^n}{n+1} = 0$.

Continuity and Convergence Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$

Definition A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is called monotonic if it is either increasing or decreasing.

Definition A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \leq M$ for all $n \geq 1$.

A sequence $\{a_n\}$ is bounded below if there is a number m such that $a_n \geq m$ for all $n \geq 1$.

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Monotonic Sequence Theorem Every bounded monotonic sequence is convergent.

Section 8.2 Series

Example 1. $\sqrt{2} = 1.41421356... = 1 + 4 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10^2} + 4 \cdot \frac{1}{10^3} + 2 \cdot \frac{1}{10^4} + 1 \cdot \frac{1}{10^5} + 3 \cdot \frac{1}{10^6}$
$+5 \cdot \frac{1}{10^7} + 6 \cdot \frac{1}{10^8} + \cdots$
It is an infinite sum (series) of numbers.

In general, if we have a sequence $\{a_n\}_{n=1}^{\infty}$ then we may want to add its terms:

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{n=1}^{\infty} a_n$$

This expression is called an infinite series (or just a series).

To define a value of a series consider its partial sums:

$$s_1 = a_1,\;\; s_2 = a_1 + a_2,\;\; s_3 = a_1 + a_2 + a_3,\; ...,\;\; s_n = a_1 + a_2 + \cdots a_n = \sum_{i=1}^{n} a_i,$$

where s_n is called nth partial sum. The partial sums form a new sequence $\{s_n\}_{n=1}^{\infty}$

If the sequence $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = s$, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and $\sum_{n=1}^{\infty} a_n = s$. The number s is called the sum of the series.

If $\{s_n\}_{n=1}^{\infty}$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is called divergent.

Special notation: $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n,\; a \neq 0.$

It is convergent when $|r| < 1$. In this case its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$

If $|r| \geq 1$, then the geometric series is divergent.

Example 2. Represent the number $3.\overline{46}$ as a ratio of integers.

Solution: $3.\overline{46} = 3.46464646... = 3 + \frac{46}{100} + \frac{46}{100^2} + \frac{46}{100^3} + \frac{46}{100^4} + \cdots$

$$= 3 + \frac{46}{100}\left(1 + \frac{1}{100^2} + \frac{1}{100^3} + \frac{1}{100^4} + \cdots\right) = 3 + \sum_{n=0}^{\infty} \frac{46}{100}\left(\frac{1}{100}\right)^n.$$

It is a geometric series with $a = \dfrac{46}{100}$ and $-1 < r = \dfrac{1}{100} < 1$.

Therefore, $3.\overline{46} = 3 + \dfrac{46/100}{1 - 1/100} = 3 + \dfrac{46/100}{99/100} = 3 + \dfrac{46}{99} = \dfrac{297 + 46}{99} = \dfrac{343}{99}$

Example 3. $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$.

Example 4. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ is convergent and find its limit.

Solution: $\dfrac{2}{n(n+2)} = \dfrac{1}{n} - \dfrac{1}{n+2}$.

The partial sum is

$$\sum_{i=1}^{n} \frac{2}{n(n+2)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+2}\right)$$

$$= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n(n+2)} = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}$.

Harmonic series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Test for divergence If $\displaystyle\lim_{n\to\infty} a_n$ DNE or $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$ is divergent.

Solution: $\displaystyle\lim_{n\to\infty} \frac{3n^2}{n(n+2)} = 3 \neq 0$. By the test for divergence the series is divergent.

Section 8.3 The Integral and Comparison Tests

The Integral Test Suppose $a_n = f(n)$, $n \geq n_0$ where f is a continuous, positive function on $[n_0, \infty)$ decreasing on $[m, \infty)$ for some $m \geq n_0$. Then the series $\sum_{n=n_0}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{n=n_0}^{\infty} f(x)\,dx$ is convergent. In other words:

If $\int_{n=n_0}^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=n_0}^{\infty} a_n$ is convergent.

If $\int_{n=n_0}^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=n_0}^{\infty} a_n$ is divergent.

Example 1. Using the Integral Test determine whether the series $\sum_{n=3}^{\infty} \frac{2n-3}{n(n-3)}$ is convergent or divergent.

Solution: Consider the function $f(x) = \frac{2x-3}{x(x-3)} = \frac{2x-3}{x^2-3x}$. Define its domain as $x \geq n_0 = 4$ on which f is continuous and positive. Is it decreasing there? Take its derivative

$$f'(x) = \frac{2(x^2-3x)-(2x-3)^2}{(x^2-3x)^2} = \frac{2x^2-6x-4x^2+12x-9}{(x^2-3x)^2} = \frac{-2x^2+6x-9}{(x^2-3x)^2}$$

The denominator is always positive inside the domain. The descriminant of the numerator is $D = 9 - 18 = -9 < 0$. When $x = 0$ the numerator is $-9 < 0$. Hence, it is negative for $x \geq 4$. So, the function is decreasing.

$$\int_4^{\infty} f(x)\,dx = \int_4^{\infty} \frac{2x-3}{x^2-3x}\,dx = \lim_{t\to\infty} \int_4^{t} \frac{2x-3}{x^2-3x}\,dx$$

u-sub: $u = x^2 - 3x,$, $du = (2x-3)\,dx$, $u(4) = 4$, $u(t) = t^2 - 3t$.

$$= \lim_{t\to\infty} \int_4^{t^2-3t} \frac{du}{u} = \lim_{t\to\infty} \Big[\ln u\Big]_4^{t^2-3t} = \infty$$

So, the integral is divergent and hence the series is divergent.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series. It is convergent when $p > 1$ and divergent when $p \leq 1$.

Example 2. Show that the series p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Solution: We apply the Integral Test and consider the function $f(x) = \dfrac{1}{x^2}$ with the domain defined as $x \geq 1$. On the domain f is continuous, positive, and decreasing.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = 1$$

So, the integral is convergent and hence the series is convergent.

The Copmarison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with $0 \leq a_n \leq b_n$ for all n. Then

If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Example 3. Using the Comparison Test determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{2n}{(n+1)^3}$ is convergent or divergent.

Solution: Consider the function $0 \leq \dfrac{2n}{(n+1)^3} \leq \dfrac{2n}{n^3} = 2 \cdot \dfrac{1}{n^2}$. The last series is the p-series with $p = 2$ and hence it is convergent. By the Comparison Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+1)^3}$ is convergent an so is $\displaystyle\sum_{n=5}^{\infty} \frac{2n}{(n+1)^3}$.

The Limit Copmarison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$ is a finite number, then either both series converge or both diverge.

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5+2^n}{3^n}$ is convergent or divergent.

Solution: Consider the geometric series $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ which is convergent because $r = \dfrac{2}{3} < 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5+2^n}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{5+2^n}{2^n} = 1.$$

By the Limit Copmarison Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{5+2^n}{3^n}$ is convergent.

Section 8.4 Other Convergence Tests

The Alternating Series Test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\, b_n,\ \ b_n > 0$ satisfies

$$(1)\quad b_{n+1} \leq b_n \qquad \text{for all } n$$
$$(2)\quad \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

Example 1. Determine whether the series $\sum_{n=3}^{\infty}(-1)^{n-1}\sin\left(\frac{\pi}{n}\right)$ is convergent or divergent.

Solution: It is an alternating series (note $\sin\left(\frac{\pi}{n}\right) > 0$, when $n \geq 3$). $\sin x$ is increasing function on $[0, \pi/3]$. Hence, $\sin\left(\frac{\pi}{n+1}\right) < \sin\left(\frac{\pi}{n}\right)$ and the condition (1) holds.

Also, $\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right) = 0$ and the condition (2) holds.

By the alternating series test the series is convergent.

Alternating Series Estimation Theorem If $s = \sum_{n=1}^{\infty}(-1)^{n-1}\, b_n$ is the sum of the alternating series that satisfies

$$(1)\quad 0 \leq b_{n+1} \leq b_n \qquad \text{for all } n$$
$$(2)\quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example 2. How many terms of the series $\sum_{n=3}^{\infty}(-1)^{n-1}\sin\left(\frac{\pi}{n}\right)$ we need to add in order to find the sum to the accuracy $|error| < 0.003$?

Solution: The condition $|error| < 0.003$ is equivalent $|b_{n+1}| = \sin\left(\frac{\pi}{n+1}\right) < 0.003$. Then we get $\frac{\pi}{n+1} < sin^{-1}(.003) = 0.003$. We need to solve this inequality for n?: $n + 1 < \frac{\pi}{0.003} =$ 1047.2. Hence, $n + 1 \leq 1047$ and $n \leq 1046$. Answer: 1046 terms.

Absolute Convergence A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent.

A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example 3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-1}$ is absolutely convergent, conditionally convergent, or divergent.

Solution: It is an alternating series. Denote $b_n = \dfrac{1}{3n-1}$. Then $b_n > 0$ if $n \geq 1$, $b_{n+1} = \dfrac{1}{3n+2} < \dfrac{1}{3n-1} = b_n$, and $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n-1} = 0$. By the alternating series test the given series is convergent.

Now consider $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{3n-1}\right| = \sum_{n=1}^{\infty} \frac{1}{3n-1}$.

$\dfrac{1}{3n-1} = \dfrac{1}{3} \cdot \dfrac{1}{n-1/3} > \dfrac{1}{3} \cdot \dfrac{1}{n}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent harmonic series. Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n-1}$ is divrgent.

The given series is conditionally convergent.

The Ratio Test

(1) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is absolutely convergent.

(2) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum a_n$ is divergent.

(2) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive.

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{(3n-1)!}$ is convergent.

Solution: $a_n = \dfrac{(-5)^n}{(3n-1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5^{n+1}}{(3n+2)!} \cdot \frac{(3n-1)!}{5^n} = \lim_{n\to\infty} \frac{5}{3n(3n+1)(3n+2)} = 0 < 1.$$

By The Ratio test the series is absolutely convergent, and hence convergent.

The Root Test

(1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum a_n$ is divergent.

(2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

Example 5. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^2+n}{3n^2-1}\right)^n$ is convergent.

Solution: $\quad a_n = \left(\frac{2n^2+n}{3n^2-1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n^2+n}{3n^2-1} = \frac{2}{3} < 1.$$

By The Root test the series is absolutely convergent, and hence convergent.

Section 8.5 Power Series

A Power Series centered at a is a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$

(Here we assume that $(x-a)^0 = 1$ even when $x = a$).

Example 1. For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

Solution: We use the Ratio test: $a_n = n!\, x^n$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} (n+1)\ |x| = \infty \text{ if } x \neq 0.$$

Hence, the series is convergent only when $x = 0$.

Example 2. For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n\, n^3}$ convergent?

Solution: The Ratio test: $a_n = \frac{(x-5)^n}{2^n\, n^3}$,

$$\lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{2^{n+1}\,(n+1)^3} \cdot \frac{2^n\, n^3}{(x-5)^n} \right| = \lim_{n\to\infty} \frac{n^3}{2(n+1)^3} \cdot |x-5| = \frac{|x-5|}{2} < 1,\ |x-5| < 2.$$

Hence, the series is convergent only when $|x-5| < 2$ or $3 < x < 7$.

If $x = 3$ or $x = 7$, the Ratio test gives no information. Consider these cases separately.

$x = 3$: $x - 5 = -2 = (-1) \cdot 2$. The series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{2^n\, n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$.

It is an alternating decreasing series with $b_n = \frac{1}{n^3}$ and $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

By the Alternating Series test it is convergent.

$x = 7$: $x - 5 = 2$. The series becomes $\sum_{n=1}^{\infty} \frac{2^n}{2^n\, n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

It is the p-series with $p = 3 > 1$ and is convergent.

Therefore, the given series is convergent when $3 \leq x \leq 7$.

For a given **power series** $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilieties:

(1) the series converges only when $x = a$.

(2) the series converges for all x.

(3) there is a positive number R such that the series converges if $|x-a| < R$ and diverges if

$|x-a|>R$.

In the last case $(a-R, a+R)$ is called the interval of convergence and R is called the radius of convergence. An addition work is required to determine if the endpoints $a-R$ and $a+R$ of the interval of convergence are included.

This is how R can be found: By the Ratio test the inequality

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \cdot |x-a| < 1 \text{ must hold.}$$

It is equivalent to the inequality $|x-a| < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = R$.

Example 3. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty}\frac{(2x+3)^n}{5^n}$.

Solution: $\sum_{n=0}^{\infty}\frac{(2x+3)^n}{5^n} = \sum_{n=0}^{\infty}\frac{2^n(x+3/2)^n}{5^n} = \sum_{n=0}^{\infty}\frac{2^n}{5^n}\cdot(x+3/2)^n, \quad c_n = \frac{2^n}{5^n},$

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{2^n}{5^n}\cdot\frac{5^{n+1}}{2^{n+1}} = \lim_{n\to\infty}\frac{5}{2} = \frac{5}{2}.$$

Hence, the series is convergent only when $\left|x+\frac{3}{2}\right| < \frac{5}{2}$ or $-4 < x < 1$.

$x=-4$: $\frac{(2x+3)^n}{5^n} = \frac{(-5)^n}{5^n} = (-1)^n$. The series becomes $\sum_{n=0}^{\infty}(-1)^n$ which is divergent.

$x=1$: $\frac{(2x+3)^n}{5^n} = \frac{5^n}{5^n} = 1$. The series becomes $\sum_{n=0}^{\infty}1$ which is divergent.

Therefore, the radius of convergence is $R=\frac{5}{2}$ and interval of convergence is $(-4,1)$.

Section 8.6 Representing Functions as Power Series

Example 1. $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n, \ \ |x| < 1.$

Example 2. Find a power series representation and the interval of convergence for $\dfrac{1}{5x+3}$.

Solution: $$\frac{1}{5x+3} = \frac{1}{3+5x} = \frac{1}{3\left(1+\frac{5}{3}x\right)} = \frac{1}{3}\cdot\frac{1}{1-\left(-\frac{5}{3}x\right)}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{5}{3}x\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{5^n}{3^{n+1}}x^n, \qquad c_n = (-1)^n\frac{5^n}{3^{n+1}}.$$

The radius of convergence is $R = \displaystyle\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{5^n}{3^{n+1}}\cdot\frac{3^{n+2}}{5^{n+1}} = \lim_{n\to\infty}\frac{3}{5} = \frac{3}{5}.$

The interval of convergence is $\left(-\dfrac{3}{5}, \dfrac{3}{5}\right)$.

Example 3. Find a power series representation and the interval of convergence for $\dfrac{x}{x^2+4}$.

Solution: $$\frac{x}{x^2+4} = \frac{x^2}{4+x^2} = \frac{x^2}{4\left(1+\frac{1}{4}x^2\right)} = \frac{x^2}{4}\cdot\frac{1}{1-\left(-\frac{1}{4}x^2\right)}$$

$$= \frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n = -\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^{n+1}.$$

It is a geometric series and it converges when $\left|-\dfrac{x^2}{4}\right| < 1$, or $x^2 < 4$, or $|x| < 2$.

The interval of convergence is $(-2, 2)$.

Differentiation and Integration of Power Series

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then

(1) $\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(2) $\displaystyle\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the powers in (1) and (2) are both R.

Example 4. Find a power series representation for $\dfrac{1}{(1-x)^2}$.

Solution: $$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} nx^{n-1}$$

Example 5. Find a power series representation for $\ln(5x+3)$ and its radius of convergence.

Solution: $\ln(5x+3) = 5\int \frac{1}{5x+3}\,dx = \int \left[\sum_{n=0}^{\infty}(-1)^n \frac{5^{n+1}}{3^{n+1}}\,x^n\right]$

$= C + \sum_{n=0}^{\infty}(-1)^n \frac{5^{n+1}}{3^{n+1}(n+1)}\,x^{n+1} = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{5^n}{3^n\,n}\,x^n.$

$c_n = (-1)^{n-1}\frac{5^n}{3^n\,n}.$

The radius of convergence is $R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{5^n}{3^n\,n}\cdot\frac{3^{n+1}(n+1)}{5^{n+1}} = \lim_{n\to\infty}\frac{3}{5}\cdot\frac{n+1}{n} = \frac{3}{5}.$

Section 8.7 Taylor and Maclaurin Series

Taylor Series If f has a power series representation at a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ |x-a| < R, \text{ then } c_n = \frac{f^{(n)}(a)}{n!}. \text{ So,}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin Series The Taylor series of f centerd at 0 is called Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Example 1. Find the Maclaurin series for $f(x) = \sin x$.

Solution: We need to find coefficients $c_n = \dfrac{f^{(n)}(0)}{n!}$:

$$c_0 = f(0) = \sin 0 = 0, \ c_1 = f'(0) = \cos 0 = 1, \ c_2 = \frac{f''(0)}{2!} = \frac{-\sin 0}{2} = 0,$$

$$c_3 = \frac{f'''(0)}{3!} = \frac{-\cos 0}{3!} = -\frac{1}{3!}, \ c_4 = \frac{f^4(0)}{4!} = \frac{\sin 0}{4!} = 0, \ c_5 = \frac{f^5(0)}{5!} = \frac{\cos 0}{120} = \frac{1}{5!}, \ ...$$

Hence, $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

The nth-degree Taylor polynomial of f at a is $T_n(x) = \displaystyle\sum_{i=0}^{n} c_i (x-a)^i$.

Then $f(x) = \displaystyle\lim_{n\to\infty} T_n(x)$.

The remainder of the Taylor series is $R_n(x) = f(x) - T_n(x)$.

Taylor's Formula $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$, for some z strictly between x and a.

Example 2. Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution: $f(x) = (1+x)^k$, $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$, $f(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$, $f(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$, $f(0) = k(k-1)(k-2)$

$\cdots$

$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$

We obtain

The Binomial Series $\quad (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$

where $\binom{k}{n} = \dfrac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$, $\quad k$ is any real number and $|x| < 1$.

Example 3. Use the Binomial series to expand the function $f(x) = \dfrac{1}{\sqrt[3]{8-x}}$ as a power series. Find its radius of convergence.

Solution: $\quad f(x) = (8-x)^{-1/3} = \left(8\left(1-\dfrac{x}{8}\right)\right)^{-1/3} = \dfrac{1}{2}\cdot\left(1-\dfrac{x}{8}\right)^{-1/3}.$

Using the Binomial series with $k = -\dfrac{1}{3}$ and with x replaced by $-\dfrac{x}{8}$, we have

$$\frac{1}{\sqrt[3]{8-x}} = \frac{1}{2}\cdot\left(1-\frac{x}{8}\right)^{-1/3} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/3}{n}\left(-\frac{x}{8}\right)^n$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{8}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{8}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{8}\right)^3\right.$$

$$\left. + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{8}\right)^n\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{16}\,x + \frac{1\cdot 3}{2!\,16^2}\,x^2 + \frac{1\cdot 3\cdot 5}{3!\,16^3}\,x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!\,16^n}\,x^n + \cdots\right]$$

The series is convergent when $\left|-\dfrac{x}{8}\right| < 1$, that is, $|x| < 8$. So the radius of convergence is $R = 8$.

Several Special Series see page 484.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and etc.

Example 4. Find the Maclaurin series for $f(x) = e^x \cos x$.

Solution: $$f(x) = \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right]\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right]$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^4}{24} + \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

Section 8.8 Applications of Taylor Polynomials

Recall

The nth-degree Taylor polynomial of f at a is $T_n(x) = \sum_{i=0}^{n} c_i (x-a)^i$.

Then $f(x) = \lim_{n\to\infty} T_n(x)$.

Approximating Functions by Taylor Polynomials $\quad f(x) \approx T_n(x)$.

The remainder of the Taylor series is $R_n(x) = f(x) - T_n(x)$.

Taylor's Formula $\quad R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$, for some z strictly between x and a.

Alternating Series Estimation Theorem $\quad |R_n| = |s - s_n| \le b_{n+1}$

Example 1. (a) What is the maximum error possible in using the approximation

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

to find $\cos\dfrac{\pi}{20}$ when $-0.2 \le x \le 0.2$? Use this approximation to find $\cos\dfrac{\pi}{20}$ correct to five decimal places. Notice, that $\dfrac{\pi}{20} < 0.2$.

(b) For what values of x is this approximation accurate to within 0.0001?

Solution: (a) We can apply the alternating series estimation theorem if we consider the given series as an alternating series with $b_1 = 1$, $b_2 = \dfrac{x^2}{2}$, $b_3 = \dfrac{x^4}{24}$, $b_4 = \dfrac{x^6}{720}$. Then $|R_3| \le b_4$.

If $-0.2 \le x \le 0.2$, then the maximum value for b_4 is $\dfrac{(0.2)^6}{720} \approx 0.000000089 = 8.9 \times 10^{-8}$.

Hence the error is smaller than 8.9×10^{-8} and the approximation gives an estimate correct to five decimal places.

The estimate is $\quad \cos x \approx 1 - \dfrac{\left(\frac{\pi}{20}\right)^2}{2} + \dfrac{\left(\frac{\pi}{20}\right)^4}{24} = 0.98769.$

(b) The error is smaller than 0.0001 if $b_4 = \dfrac{x^6}{720} < 0.0001$.

We have to solve this inequality: $x^6 < 720 \cdot 0.0001$, $x^6 < 0.072$, $|x| < \sqrt[6]{0.072} \approx 0.6449936$.

So the given approximation is accurate to within 0.0001 when $|x| < 0.6449936$.

If we used the Taylor's formula then we would get $T_4(x) = T_5(x)$, $R_5(x) = \dfrac{f^{(6)}(z)}{6!}x^6$,

where $|f^{(6)}(z)| \le 1$, since $|f^{(6)}(z)|$ is either $|\cos z|$ or $|\sin z|$.

Then the Taylor's formula gives the same estimate for the error as the alternating series does:

$|R_5(x)| = \dfrac{|f^{(6)}(z)|x^6}{6!} \le \dfrac{x^6}{720}$.