Math 0230 Calculus 2 Lectures

Chapter 9 Parametric Equations and Polar Coordinates

Numeration of sections corresponds to the text

James Stewart, Essential Calculus, Early Transcendentals, Second edition.

Section 9.1 Parametric Curves

Parametric equations: $x = f(t)$, $y = g(t)$, where $f(t)$ and $g(t)$ are continuous functions. They define a curve in the xy-plane when t runs along an interval $[a, b]$: $(x, y) = (f(t), g(t))$, when $a \le t \le b$. Initial point of the curve is $(f(a), g(a))$ and terminal point is $(f(b), g(b))$. In the parametric form y is not necessarily must be a function of x.

Example 1. Sketch the curve $x = t - 1$, $y = t^2 - 3$, when $-2 \le t \le 4$.

Solution: It is a parabola:

t	x	y
-2	-3	1
-1	-2	-2
0	-1	-3
1	0	-2
2	1	1
3	2	6
4	3	13

Sketch its graph.
Algebraic justification that it ia a parabola: $t = x + 1$, $y = (x + 1)^2 - 3 = x^2 + 2x - 2$.

Example 2. (a) Sketch the curve $x = \cos t$, $y = \sin t$, when $0 \le t \le 2\pi$.

Solution: It is the unit circle: $x^2 + y^2 = 1$

(b) Sketch the curve $x = h + r\cos t$, $y = k + r\sin t$, when $0 \le t \le 2\pi$.

Solution: It is a circle centerd at (h, k) with radius r:

$(x - h)^2 + (y - k)^2 = r^2\cos^2 t + r^2\sin^2 t = r^2$.

Section 9.2 Calculus with Parametric Curves

Tangents

$x = f(t)$, $y = g(t)$. Here we assume that $f(t)$ and $g(t)$ are differentiable functions. y can be expressed as $y = x(t)$. Using the Chain Rule we get

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}.$$

The derivative evaluated at a point (x_0, y_0), such that $x_0 = x(t_0)$, $y_0 = y(t_0)$ for some t_0, is a slope of the tangent line to the parametric curve at that point.

If $\frac{dy}{dt} = 0$, then the tangent is horizontal. If $\frac{dx}{dt} = 0$, then the tangent is vertical.

For the second derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example 1. The curve $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$ is an ellipse.

(a) Find an equation of the tangent line to the ellipse at a point where $t = \pi/3$.
(b) Find all points on the ellipse at which the tangent line is vertical.

Solution: (a) Slope $m = \dfrac{dy/dt}{dx/dt}$ when $t = \pi/3$.

$$\left.\frac{dy}{dt}\right|_{t=\pi/3} = 3\cos t \left.\vphantom{\frac{dy}{dt}}\right|_{t=\pi/3} = 3\cos(\pi/3) = 3/2$$

$$\left.\frac{dx}{dt}\right|_{t=\pi/3} = -2\sin t \left.\vphantom{\frac{dx}{dt}}\right|_{t=\pi/3} = -2\sin(\pi/3) = -\sqrt{3}$$

So, $m = -\dfrac{3}{2\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$

The point on the curve is $(2\cos\pi/3, 3\sin\pi/3) = \left(1, \dfrac{3\sqrt{3}}{2}\right)$.

The point slope form of the tangent line is $y = \dfrac{3\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2}(x-1)$ or $y = -\dfrac{\sqrt{3}}{2}x + 2\sqrt{3}$

(b) $\dfrac{dx}{dt} = 0$, $-2\sin t = 0$, $t = 0$, $t = \pi$.

Areas

The area under the curve given by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ is

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt$$

or

$$A = \int_{\beta}^{\alpha} g(t) f'(t)\, dt$$

if $(f(\beta), g(\beta))$ is the most left endpoint.

Example 2. Find the area enclosed by the ellipse $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$.

Solution: $A = 2\int_{\pi}^{0} 3\sin t(-2\sin t)\, dt = 12\int_{0}^{\pi} \sin^2 t\, dt = 6\int_{0}^{\pi} (1 - \cos 2t)\, dt$

$= 6\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi} = 6\,[\pi - 0] = 6\pi$

Arc Length

The length L of the curve $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 3. Find the length L of the unit circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

Solution: $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$. Then,

$$L = \int_{0}^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_{0}^{2\pi} dt = 2\pi$$

Section 9.3 Polar Coordinates

Polar vs Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, $\tan\theta = \dfrac{y}{x}$.

Example 1. Convert the point $(4, \pi/6)$ from polar to Cartesian coordinates.

Solution: $r = 4$, $\theta = \pi/6$, $x = 4\cos\pi/6 = 4 \cdot \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$, $y = 4\sin\pi/6 = 4 \cdot \dfrac{1}{2} = 2$, $(2\sqrt{3}, 2)$.

Example 2. Convert the points

(a) $(1, \sqrt{3})$

(b) $(1, -\sqrt{3})$

(c) $(-1, \sqrt{3})$

(d) $(-1, -\sqrt{3})$

from Cartesian to polar coordinates.

Solution: $x = \pm 1$, $y = \pm\sqrt{3}$, $r = \sqrt{1+3} = 2$.

(a) $\tan\theta = \sqrt{3}$, $\theta = \pi/3$. The point is $\left(2, \dfrac{\pi}{3}\right)$.

(b) $\tan\theta = -\sqrt{3}$, $\theta = -\pi/3$. The point lies in the 4th quadrant: $\left(2, -\dfrac{\pi}{3}\right) = \left(2, \dfrac{5\pi}{3}\right)$.

(c) $\tan\theta = -\sqrt{3}$, $\theta = 2\pi/3$. The point lies in the 2nd quadrant: $\left(2, \dfrac{2\pi}{3}\right)$.

(b) $\tan\theta = \sqrt{3}$, $\theta = 4\pi/3$. The point lies in the 3rd quadrant: $\left(2, \dfrac{4\pi}{3}\right)$.

Polar Curves

General equation either $r = r(\theta)$ or $F(r, \theta) = 0$.

Example 3. Sketch the curve $r = 1$.

Solution: $x^2 + y^2 = 1$, it is the unit circle.

Example 4. Sketch the curve $\theta = \pi/4$.

Solution: $\dfrac{y}{x} = \tan(\pi/4) = 1$, $y = x$, it is a line.

Example 5. Sketch the curve with the polar equation $r = 4\sin\theta$.

Solution: $\sin\theta = \dfrac{r}{4}$, $y = r\sin\theta = \dfrac{r^2}{4}$, $4y = r^2 = x^2 + y^2$, $x^2 + y^2 - 4y = 0$, $x^2 + y^2 - 4y + 4 = 4$, $x^2 + (y-2)^2 = 2^2$, it is a circle with center at $(0, 2)$ and radius 2.

In general. Consider the polar equation $r = a\cos\theta$, where a is a positive constant. Multiplying both sides by r, we obtain $r^2 = ar\cos\theta$, which in rectangular coordinates becomes $x^2 + y^2 = ax$ or $(x^2 - ax) + y^2 = 0$. Next we complete the square on the terms in parentheses by adding $a2/4$ to both sides, obtaining $(x - a/2)^2 + y^2 = (a/2)^2$. Now we recognize this as the equation of a circle centered at the point $(a/2, 0)$ with radius $a/2$. Similarly, the equation $r = a\sin\theta$ represents a circle centered at the point $(0, a/2)$ with radius $a/2$. Note that both of these circles are traversed twice as θ varies from 0 to 2π.

Tangents to Polar Curves

$r = r(\theta)$, $x = r\cos\theta = r(\theta)\cos\theta$, $y = r\sin\theta = r(\theta)\sin\theta$.

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta,\ \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta,\ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

If $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$, then the tangent is vertical.

Example 6. Find the slope of the tangent to the curve $r = 3 - \cos\theta$ at the point where $\theta = \pi/4$.

Solution: $m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ when $\theta = \pi/4$

$$\left.\frac{dr}{d\theta}\right|_{\theta=\pi/4} = \sin\theta \left.\right|_{\theta=\pi/4} = \sin(\pi/4) = \frac{\sqrt{2}}{2},\quad r(\pi/4) = 3 - \frac{\sqrt{2}}{2}.$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/4} = \frac{\sqrt{2}}{2}\cos(\pi/4) - \left(3 - \frac{\sqrt{2}}{2}\right)\sin(\pi/4) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(3 - \frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}$$

$$= 1 - \frac{3\sqrt{2}}{2} = \frac{2 - 3\sqrt{2}}{2}$$

$$\left.\frac{dy}{d\theta}\right|_{\theta=\pi/4} = \frac{\sqrt{2}}{2}\sin(\pi/4) + \left(3 - \frac{\sqrt{2}}{2}\right)\cos(\pi/4) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} + \left(3 - \frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$

$$m = \frac{3\sqrt{2}}{2}\cdot\frac{2}{2 - 3\sqrt{2}} = \frac{3\sqrt{2}}{2 - 3\sqrt{2}}\cdot\frac{2 + 3\sqrt{2}}{2 + 3\sqrt{2}} = \frac{6\sqrt{2} + 18}{-14} = -\frac{3\sqrt{2} + 9}{7}$$

Section 9.4 Areas and Lengths in Polar Coordinates

Areas

Let R be a region bounded by the polar curves $r = r(\theta)$, $\theta = a$, and $\theta = b$. Its area is

$$A = \frac{1}{2} \int_a^b r^2(\theta)\, d\theta$$

If R is a region bounded by the polar curves $r = r_1(\theta)$, $r = r_2(\theta)$, $\theta = a$, and $\theta = b$, where $r_1(\theta) \leq r_2(\theta)$, then its area is

$$A = \frac{1}{2} \int_a^b \left(r_2^2(\theta) - r_1^2(\theta)\right)\, d\theta$$

Example 1. Find the area enclosed by one loop of the three-leaved rose $r = \cos 3\theta$.

Solution: $-\pi/6 \leq \theta \leq \pi/6$. Hence,

$$A = \frac{1}{2} \int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta = \int_0^{\pi/6} \cos^2(3\theta)\, d\theta = \frac{1}{2} \int_0^{\pi/6} (1 + \cos 6\theta)\, d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{1}{6} \sin 6\theta\right]_0^{\pi/6} = \frac{1}{2} \left[\frac{\pi}{6} + 0\right] = \frac{\pi}{12}$$

Example 2. Find the area that lies inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$.

Solution: The first curve is a circle centered at the point $(3/2, 0)$ with radius $3/2$. The second curve is a cardioid. Their points of intersections: $3\cos\theta = 1 + \cos\theta$, $\cos\theta = 1/2$, $\theta = -pi/3$ and $\theta = \pi/3$. Hence,

$$A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(9\cos^2\theta - (1 + \cos\theta)^2\right)\, d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta\right)\, d\theta$$

$$= \int_0^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta\right)\, d\theta = \int_0^{\pi/3} (4 + 4\cos 2\theta - 1 - 2\cos\theta)\, d\theta$$

$$= \int_0^{\pi/3} (4\cos 2\theta + 3 - 2\cos\theta)\, d\theta = \left[2\sin 2\theta + 3\theta - \ 2\sin\theta\right]_0^{\pi/3} = 3 \cdot \frac{\pi}{3} = \pi$$

Arc Length

Let $r = r(\theta)$, $a \le \theta \le b$ be a polar curve. $r = r(\theta)$, $\theta = a$, and $\theta = b$. We regard θ as a parameter.

Then $x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$, $\dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta$, $\dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta$ and

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

And the length of the curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 3. Find the lenght of the curve $r = 2\cos\theta$, $0 \le \theta \le \pi/6$.

Solution: $\dfrac{dr}{d\theta} = -2\sin\theta$. Hence,

$$L = \int_0^{\pi/6} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta = 2\int_0^{\pi/6} d\theta = \frac{\pi}{3}$$

Example 4. [Hard] Find the lenght of the cardioid $r = 1 - \cos\theta$.

Solution: $0 \le \theta \le 2\pi$. Hence,

$$L = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = 8$$

Example 5. [Hard] Find the lenght of one loop of the three-leaved rose $r = \cos 3\theta$.

Solution: $-\pi/6 \le \theta \le \pi/6$. Hence,

$$L = \int_{-\pi/6}^{\pi/6} \sqrt{(\cos 3\theta)^2 + (-3\sin 3\theta)^2}\, d\theta = 2\int_0^{\pi/6} \sqrt{\cos^2 3\theta + 9\sin^2 3\theta}\, d\theta$$