Midterm Exam 2

Spring 2012 Math 0280

100 points total **Your name:** Solutions

No calculators. Show all your work (no work = no credit). Write neatly.

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1. (15 points) Prove that every line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

Any line ℓ in $\mathbb{R}^2$ through the origin has the equation $y = mx$ for some scalar m.

Let $\bar{v} \in \ell$, then $\bar{v} = \begin{bmatrix} x_1 \\ mx_1 \end{bmatrix}$

$\bar{u} \in \ell$, then $\bar{u} = \begin{bmatrix} x_2 \\ mx_2 \end{bmatrix}$

$$\bar{v} + \bar{u} = \begin{bmatrix} x_1 + x_2 \\ mx_1 + mx_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ m(x_1 + x_2) \end{bmatrix} \in \ell$$

$$c\bar{v} = c\begin{bmatrix} x_1 \\ mx_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cmx_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ m \cdot cx_1 \end{bmatrix} \in \ell$$

Hence ℓ is a subspace of $\mathbb{R}^2$.

2. (15 points) Prove that T is a linear transformation if

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix}.$$

You can apply any method.

Let $\bar{u} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}$. Then

$$T(\bar{u}+\bar{v}) = T\begin{bmatrix} X_1+X_2 \\ Y_1+Y_2 \end{bmatrix} = \begin{bmatrix} -(Y_1+Y_2) \\ (X_1+X_2)+2(Y_1+Y_2) \\ 3(X_1+X_2)-4(Y_1+Y_2) \end{bmatrix}$$

$$= \begin{bmatrix} -Y_1+(-Y_2) \\ (X_1+2Y_1)+(X_2+2Y_2) \\ (3X_1-4Y_1)+(3X_2-4Y_2) \end{bmatrix} = \begin{bmatrix} -Y_1 \\ X_1+2Y_1 \\ 3X_1-4Y_1 \end{bmatrix} + \begin{bmatrix} -Y_2 \\ X_2+2Y_2 \\ 3X_2-4Y_2 \end{bmatrix}$$

$$= T\bar{u} + T\bar{v}$$

$$T(c\bar{u}) = T\begin{bmatrix} cX_1 \\ cY_1 \end{bmatrix} = \begin{bmatrix} -cY_1 \\ cX_1+2cY_1 \\ 3cX_1-4cY_1 \end{bmatrix} = \begin{bmatrix} c(-Y_1) \\ c(X_1+2Y_1) \\ c(3X_1-4Y_1) \end{bmatrix}$$

$$= c\begin{bmatrix} -Y_1 \\ X_1+2Y_1 \\ 3X_1-4Y_1 \end{bmatrix} = cT\bar{u}$$

Hence, T is linear transformation.

3. (20 points) By calculating $\det(A-\lambda I)$ find all the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

and an eigenvector that corresponds to the larger eigenvalue.

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4$$

$$= -2-\lambda+\lambda^2-4 = \lambda^2-\lambda-6 = 0$$

$$(\lambda+2)(\lambda-3) = 0 \Rightarrow \lambda_1 = -2, \quad \lambda_2 = 3$$

For $\lambda_2 = 3$ $\quad (A-\lambda_2 I)\bar{x} = \begin{bmatrix} 2-3 & 2 \\ 2 & -1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1+2x_2 \\ 2x_1-4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second line is a multiple of the first line. Hence $-x_1+2x_2=0$ gives all solutions or $x_1 = 2x_2$.
Take $x_2 = 1$. Then $\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an

e-vector corresponding to $\lambda_2 = 3$.

4. For the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) (10 points) find a basis for $col(A)$

$col(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$

vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are LI

(one is not a multiple of the other) and form a basis for $col(A)$.

Another method:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}}_{\text{r.r.e.f}} = R$$

First two columns of R contain leading 1s, the corresponding first two columns of A $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for $col(A)$

(b) (10 points) find a basis for $null(A)$

By the Rank Thm

$$\dim(col(A)) + \text{nullity } A = 3$$

$$2 + \text{nullity } A = 3$$

Hence $\text{nullity } A = \dim(null(A)) = 1$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$ r.r.e.f. of A

gives solutions to $A\bar{x} = 0$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$:

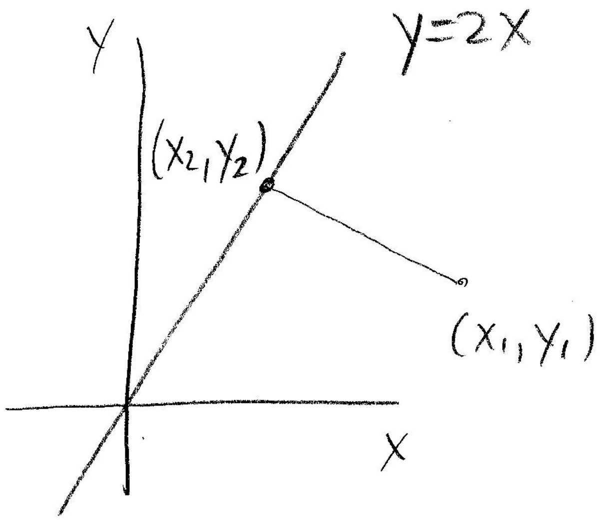
$$\begin{matrix} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \end{matrix} \quad \text{or} \quad \bar{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \bar{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow null(A) = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$$

and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a basis for $null(A)$

(one vector, $\dim(null(A)) = 1$)

5. (15 points) Find the standard matrix of the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is the projection onto the line $y = 2x$.



the line that passes through (x_1, y_1) and (x_2, y_2) has slope $-\frac{1}{2}$

Hence $\frac{y_2 - y_1}{x_2 - x_1} = -\frac{1}{2}$

$y_2 - y_1 = -\frac{1}{2}x_2 + \frac{1}{2}x_1$, $\quad y_2 = 2x_2$

$2x_2 - y_1 = -\frac{1}{2}x_2 + \frac{1}{2}x_1 \Rightarrow \frac{5}{2}x_2 = \frac{1}{2}x_1 + y_1$

$x_2 = \frac{1}{5}x_1 + \frac{2}{5}y_1$, $\quad y_2 = 2x_2 = \frac{2}{5}x_1 + \frac{4}{5}y_1$

The standard matrix is

$$\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

6. (15 points) Find a basis for the span of the given vectors from among the vectors themselves

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Form the matrix A with the given vectors as its columns:

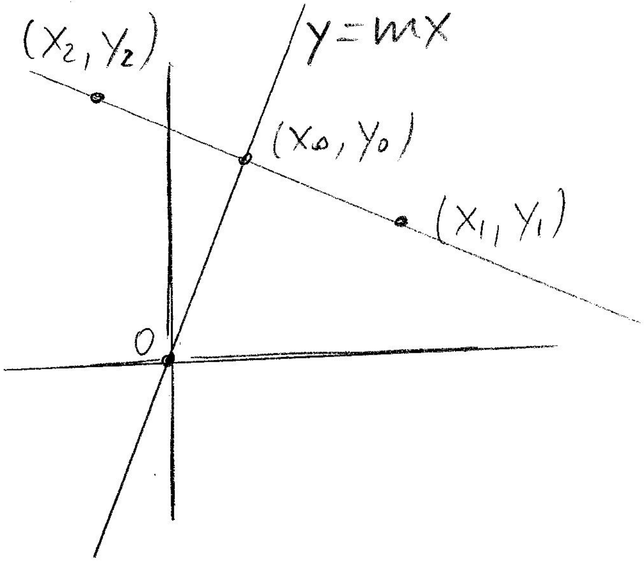
$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_3 + R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 + R_2]{R_1 + R_2} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{r.r.e.f.}}$$

$= R$

We use the columns in A that correspond to the columns in R containing leading 1s to form a basis. Hence

a basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

[Here the notation $\{\cdot,\cdot\}$ means a set of vectors] bonus problem (10 points extra) Find the standard matrix of the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is the reflection in the line $y = mx$.



Need to represent X_2 and Y_2 through X_1 and Y_1

(X_2, Y_2) is a reflection of (X_1, Y_1) in the line $Y = mX$

(X_0, Y_0) is the projection of (X_1, Y_1). From previous we know, that

$$X_0 = \frac{1}{m^2+1} X_1 + \frac{m}{m^2+1} Y_1, \quad Y_0 = \frac{m}{m^2+1} X_1 + \frac{m^2}{m^2+1} Y_1$$

(X_0, Y_0) is a midpoint of the interval that connects (X_1, Y_1) and (X_2, Y_2) $\Rightarrow X_0 = \frac{X_1 + X_2}{2}$,

$Y_0 = \frac{Y_1 + Y_2}{2}$ or $X_2 = 2X_0 - X_1$, $Y_2 = 2Y_0 - Y_1$.

The last gives

$$X_2 = \frac{1-m^2}{m^2+1} X_1 + \frac{2m}{m^2+1} Y_1$$

$$Y_2 = \frac{2m}{m^2+1} X_1 + \frac{m^2-1}{m^2+1} Y_1$$

$$A = \frac{1}{m^2+1} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$