Math 0290 **Midterm Exam 1**

Spring 2014 **S o l u t i o n s**

---

1. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

   (a) (10 points) $\dfrac{y'}{3} = x^2 y, \qquad y(0) = -8, \quad$ where $y' = \dfrac{dy}{dx}$.

   Solution: It is a separable equation

   $\dfrac{dy}{dx} = 3x^2 y, \quad \dfrac{dy}{y} = 3x^2\,dx, \quad \ln|y| = x^3 + C_1, \quad |y| = e^{C_1}\,e^{x^3}, \quad y = Ce^{x^3}, \quad C = \pm\,e^{C_1}.$

   $y(0) = C = -8, \quad y = -8e^{x^3}.$

   (b) (10 points) $ty' = 2y + t^3, \qquad y(-1) = 3.$

   Solution: It is a first order linear differential equation. $y' - \dfrac{2}{t}\,y = t^2.$

   The integrating factor is $I(t) = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = t^{-2}$. Then

   $t^{-2}y' - 2x^{-3}y = 1, \quad \left(t^{-2}y\right)' = 1, \quad tx^{-2}y = \int dt = t + C, \quad y = t^3 + Ct^2.$

   $y(-1) = -1 + C = 3, \quad C = 4, \quad y(t) = t^3 + 4t^2.$

   (c) (10 points) $y'' + 6y' + 9y = 0, \qquad y(0) = -3, \quad y'(0) = 11.$

   Solution: It is a second-order linear homogeneous differential equation.

   Its characteristic equation $r^2 + 6r + 9 = 0$ has one root $r = -3$.

   The general solution is $y(t) = e^{-3t}\,(c_1 + c_2 t)$

   Initial conditions: $y(0) = c_1 = -3,$

   $y'(t) = -3e^{-3t}(-3 + c_2 t) + c_2 e^{-3t}, \quad y'(0) = 9 + c_2 = 11, \quad c_2 = 2.$

   $y(t) = e^{-3t}\,(-3 + 2t).$

2. Suppose you drop a brick from the top of a building with the initial velocity 0 m/sec. The brick has mass of 1 kg. The air resistance force is given by $R(v) = -2v$. How long will it take the brick to reach one-half of its terminal velocity? Leave answer in exact form.

Solution: The equation $v' = -g - 2v$ is separable. Its solution is $v(t) = Ce^{-2t} - \frac{g}{2}$.

The initial condition gives $C = \frac{g}{2}$. So, $v(t) = \frac{g}{2}(e^{-2t} - 1)$. The terminal velocity is $\lim_{t\to\infty} v(t) = -\frac{g}{2}$.

We have to find time t when $v(t) = -\frac{1}{2} \cdot \frac{g}{2}$. Then,

$$\frac{g}{2}(e^{-2t} - 1) = -\frac{1}{2} \cdot \frac{g}{2}, \quad e^{-2t} - 1 = -\frac{1}{2}, \quad e^{-2t} = \frac{1}{2}, \quad -2t = -\ln 2, \quad t = \frac{\ln 2}{2} \text{ sec}$$

3. Andrew opens an account that pays an annual rate of 4% compounded continuously with no initial deposit, but agrees to deposit a fixed amount each year. What annual deposit should be made to save \$20,000 in 5 years? Simplify your answer and leave it in exact form.

Solution: The model is given by the IVP $P' = rP + D$, $P(0) = 0$.

Its solution is $P(t) = \frac{D}{r}\left(e^{rt} - 1\right)$.

We have $r = 0.04$, $t = 5$, $P(6) = 20,000$. Then $\frac{D}{0.04}\left(e^{0.04\cdot 5} - 1\right) = 20,000$,

$D = \frac{20,000 \cdot 0.04}{e^{0.2} - 1} = \frac{800}{e^{0.2} - 1}$ dollars

$[\, D = \$3,613.32 \,]$

4. (15 points) Find the general solution to the second-order nonhomogeneous differential equation

$$y'' - y' - 2y = 3\,e^{-t}$$

(a) by using the method of undetermined coefficients.

Solution: The characteristic equation of the homogeneous equation

$\lambda^2 - \lambda - 2 = 0$ has two roots $\lambda_1 = 1$ and $\lambda_2 = 2$.

The solution of the homogeneous equation is $y_h(t) = c_1 e^{-t} + c_2 e^{2t}$.

The right hand side (forcing term) is a solution to the homogeneous equation. So, for a particular solution we try $y_p(t) = ate^{-t}$.

Then $y_p{}' = a(1-t)e^{-t}$, $y_p'' = a(t-2)e^{-t}$.

Substituting into the differential equation, we have:

$a(t - 2 - 1 + t - 2t)e^{-t} = -3ae^{-t} = 3e^{-t}$ and $a = -1$

So, $y_p = -te^{-t}$

The general solution $y(t) = y_h(t) + y_p(t)$ is $y(t) = c_1 e^{-t} + c_2 e^{2t} - te^{-t} = (c_1 - t)e^{-t} + c_2 e^{2t}$

(b) by using the method of variation of parameters.

Solution: The solution of the homogeneous equation is $y_h(t) = c_1 e^{-t} + c_2 e^{2t}$. For a particular solution we try $y_p(t) = v_1(t)e^{-t} + v_2(t)e^{2t}$

Wronskian is $e^{-t} \cdot 2e^{2t} - e^{2t} \cdot (-e^{-t}) = 3e^t$

$$v_1(t) = \int \frac{-e^{2t} \cdot 3e^{-t}}{3e^t} dt = \int (-1)dt = -t, \qquad v_2(t) = \int \frac{e^{-t} \cdot 3e^{-t}}{3e^t} dt = \int e^{-3t} dt = -\frac{e^{-3t}}{3}$$

So, $y_p = -te^{-t} - \dfrac{e^{-t}}{3}$

The general solution $y(t) = y_h(t) + y_p(t)$ is $y(t) = c_1 e^{-t} + c_2 e^{2t} - te^{-t} - \dfrac{e^{-t}}{3} = (C_1 - t)e^{-t} + c_2 e^{2t}$,

where $C_1 = c_1 - 1/3$.

Both general solutions in part (a) and part (b) are the same.

bonus problem (15 points extra) Compare particular solutions found in the previous problem parts (a) and (b). Are they the same? If not, explain why both solutions are correct.

Solution: Particular solutions are different. The solution y_p in part (b) contains the term $-\dfrac{e^{-t}}{3}$ which is a part of FSS and does not affect the general solution. So, the general solutions in both parts are the same. Indeed, in part (b) the general solution can be written as

$$y(t) = c_1 e^{-t} + c_2 e^{2t} - te^{-t} - \frac{e^{-t}}{3} = \left(c_1 - \frac{1}{3}\right) e^{-t} + c_2 e^{2t} - te^{-t} = C_1 e^{-t} + c_2 e^{2t} - te^{-t} = (C_1 - t)e^{-t} + c_2 e^{2t},$$

where $C_1 = c_1 - 1/3$.