Math 0290 **Midterm Exam 1**

Fall 2014 **S o l u t i o n s**

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1. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

   (a) (15 points) $x\,y' = (1+2x^2)y, \qquad y(-1) = e, \quad$ where $y' = \dfrac{dy}{dx}$.

   Solution: It is a separable equation

   $$x\frac{dy}{dx} = (1+2x^2)y, \quad \frac{dy}{y} = \frac{1+2x^2}{x}\,dx, \quad \int \frac{dy}{y} = \int\left(\frac{1}{x}+2x\right)dx,$$

   $$\ln|y| = \ln|x| + x^2 + C, \quad |y| = |x|\,e^{x^2}\,e^C, \quad y = Axe^{x^2}, \quad A = \pm\,e^C.$$

   $$y(-1) = A(-1)e = e, \quad A = -1, \quad y(x) = -xe^{x^2}.$$

   (b) (15 points) $(t^2+2)x' + 4tx = 3, \qquad x(0) = -1.$

   Solution: Divide both sides by t^2+2 to get a first order linear differential equation:
   $$x' + \frac{4t}{t^2+2}\,x = \frac{3}{t^2+2}.$$

   The integrating factor is $I(t) = e^{\int \frac{4t}{t^2+2}\,dt} = e^{2\ln(t^2+2)} = e^{\ln(t^2+2)^2} = (t^2+2)^2$. Then

   $$(t^2+2)^2\left(x' + \frac{4t}{t^2+2}x\right) = (t^2+2)^2\cdot\frac{3}{t^2+2}, \quad (t^2+2)^2x' + 4t(t^2+2)x = 3t^2+6,$$

   $$\left((t^2+2)^2x\right)' = 3t^2+6, \quad (t^2+2)^2x = \int(3t^2+6)\,dt = t^3+6t+C, \quad x = \frac{t^3+6t+C}{(t^2+2)^2}.$$

   $$x(0) = \frac{C}{4} = -1, \quad C = -4, \quad x(t) = \frac{t^3+6t-4}{(t^2+2)^2}.$$

2. (15 points) Suppose you drop a ball from the top of a building with the initial velocity 0 m/sec. The ball has mass of 0.2 kg. The air resistance force is given by $R(v) = -\dfrac{v}{5}$.

   How long will it take the ball to reach one-half of its terminal velocity? Leave answer in exact form.

   Solution: Here $r = \dfrac{1}{5}$ and $m = 0.2 = \dfrac{1}{5}$. Then $\dfrac{r}{m} = 1$.

   The equation of the motion is either $v' = -g - v$ when the x-axis is directed up or $v' = g - v$ when the x-axis is directed down. In both cases the equation is separable or the first order linear. For $v' = -g - v$ the solution is $v(t) = Ce^{-t} - g$.

The initial condition gives $C = g$. So, $v(t) = g(e^{-t} - 1)$. The terminal velocity is $\lim_{t\to\infty} v(t) = -g$.

We have to find time t when $v(t) = -\frac{1}{2} \cdot g$. Then,

$$g(e^{-t} - 1) = -\frac{1}{2} \cdot g, \quad e^{-t} - 1 = -\frac{1}{2}, \quad e^{-t} = \frac{1}{2}, \quad -t = -\ln 2, \quad t = \ln 2 \text{ sec}$$

The answer is the same if you used the equation $v' = g - v$.

3. (15 points) Suppose the electrical circuit has a resistor of $R = 0.4\,\Omega$ and an inductor of $L = 0.2\,H$. Assume the voltage source is a constant $E = 0.6\,V$. If the initial current is $0\,A$ find the resulting current as a function of time. Simplify your answer and leave it in exact form.

Solution: The model is described by the IVP: $R\,I + L\,\frac{dI}{dt} = E, \quad I(0) = 0.$

When we plug in all constants the equation becomes $0.4I + 0.2\frac{dI}{dt} = 0.6$ or,

after multiplication by 5, $\frac{dI}{dt} + 2I = 3.$

The equation is both the first order linear and separable. Let's solve it as a separable equation:

$\frac{dI}{dt} = 3 - 2I, \quad \frac{dI}{3-2I} = dt, \quad -\frac{1}{2}\ln|3-2I| = t + C_1, \quad \ln|3-2I| = -2t + C, \quad C = -2C_1.$

Hence, $3 - 2I = Ae^{-2t}$ and $I(t) = \frac{3}{2} - \frac{1}{2}Ae^{-2t}.$

The initial condition gives $I(0) = \frac{3}{2} - \frac{1}{2}A = 0.$ Then $A = 3.$

The resulting current is $I(t) = \frac{3}{2} - \frac{3}{2}e^{-2t} = 1.5 - 1.5e^{-2t}$

(If you treat the equation as the first order linear, then the integrating factor is $u = e^{2t}$)

4. (20 points) Using the method of undetermined coefficients find the general solution of the equation

$$y'' - y' - 2y = 2e^{2t}$$

Show all the work. Mention a type of the equation.

Solution: It is a second-order linear non-homogeneous differential equation.

The characteristic equation of the corresponding homogeneous DE

$r^2 - r - 2 = 0$ has two real roots $r = -1$ and $r = 2$.

The solution of the homogeneous DE is $y_h(t) = C_1e^{-t} + C_2e^{2t}$

The right hand side of the given non-homogeneous DE coincides with one of the homogeneous solutions. Therefore to find a particular solution we have to use the form

$y_p = ate^{2t}$ (not $y_p = ae^{2t}$).

$y_p' = ae^{2t} + 2ate^{2t}, \quad y_p'' = 4ae^{2t} + 4ate^{2t}$

The left hand side of the given non-homogeneous DE is

$y'' - y' - 2y = 4ae^{2t} + 4ate^{2t} - ae^{2t} - 2ate^{2t} - 2ate^{2t} = 3ae^{2t}$.

It has to be equal the right hand side $2e^{2t}$. That gives $a = \frac{2}{3}$ and $y_p(t) = \frac{2}{3}te^{2t}$.

The general solution is $y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{2t} + \frac{2}{3}te^{2t}$.

5. (20 points) Using variation of parameters technique find a particular solution to the equation

$$y'' - 4y = e^{3t}$$

Show all the work. Mention a type of the equation.

Solution: It is a second-order linear non-homogeneous differential equation.

The characteristic equation of the corresponding homogeneous DE

$r^2 - 4 = 0$ has two real roots $r = -2$ and $r = 2$.

The fundamental set of solutions is $y_1 = e^{-2t}, \quad y_2 = e^{2t}$

The Wronskian is $W = y_1y_2' - y_1'y_2 = e^{-2t} \cdot 2e^{2t} - (-2)e^{-2t}e^{2t} = 2 + 2 = 4$

$$v_1 = \int \frac{-y_2e^{3t}}{W}\,dt = -\frac{1}{4}\int e^{2t}e^{3t}\,dt = -\frac{1}{4}\int e^{5t}\,dt = -\frac{1}{20}e^{5t}$$

$$v_2 = \int \frac{y_1e^{3t}}{W}\,dt = \frac{1}{4}\int e^{-2t}e^{3t}\,dt = \frac{1}{4}\int e^{t}\,dt = \frac{1}{4}e^{t}$$

A particular solution is $y_p = v_1y_1 + v_2y_2 = -\frac{1}{20}e^{5t}e^{-2t} + \frac{1}{4}e^te^{2t} = \frac{1}{5}e^{3t}$.

The general solution is $y(t) = C_1y_1 + C_2y_2 + y_p = C_1e^{-2t} + C_2e^{2t} + \frac{1}{5}e^{3t}$.

(If your fundamental set of solutions is $y_1 = e^{2t}, \ y_2 = e^{-2t}$ then $W = -4$ and $v_1 = \frac{1}{4}e^t$, $v_2 = -\frac{1}{20}e^{5t}$)

bonus problem (15 points extra) Find the general solution of the equation $y' = (y + t)^2$.
Hint: Use the substitution $x = y + t$

Solution: $x = y + t, \ y = x - t, \ y' = x' - 1$

So, the given equation becomes

$x' - 1 = x^2$, $\dfrac{dx}{dt} = x^2 + 1$, $\int \dfrac{dx}{x^2+1} = \int dt$, $\tan^{-1} x = t + C$, $x = \tan(t + C)$. Then $y = \tan(t + C) - t$.