Math 0290 **Midterm Exam 2**

Fall 2014 **S o l u t i o n s**

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1. Find the inverse Laplace transform of the function $F(s) = \dfrac{1}{s^2 - s}$, $s \geq 1$. Show all your work.

   Solution: $\dfrac{1}{s^2 - s} = \dfrac{1}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$

   $y(t) = L^{-1}\left[\dfrac{1}{s-1} - \dfrac{1}{s}\right] = L^{-1}\left[\dfrac{1}{s-1}\right] - L^{-1}\left[\dfrac{1}{s}\right]$, $y(t) = e^t - 1$

2. Solve the initial-value problem $y' - y = g(t)$, $y(0) = 0$, where

$$g(t) = \begin{cases} 0, & \text{for } 0 \leq t < 1 \\ 1, & \text{for } 1 \leq t < 2 \\ 0, & \text{for } t \geq 2 \end{cases}$$

   Create a piecewise definition for your solution that doesn't use the Heaviside function.

   Show all your work. You may use results from the previous problem.

   Solution: $g(t) = 0 \cdot H(t) + 1 \cdot [H(t-1) - H(t-2)] + 0 \cdot H(t-2) = H(t-1) - H(t-2)$,

   $L[y' - y] = (s-1)Y(s)$, $L[g(t)] = L\left[H(t-1) - H(t-2)\right] = \dfrac{e^{-s} - e^{-2s}}{s}$

   $(s-1)Y(s) = \dfrac{e^{-s} - e^{-2s}}{s}$,

   $Y(s) = \dfrac{e^{-s} - e^{-2s}}{s(s-1)} = (e^{-s} - e^{-2s})\left(\dfrac{1}{s-1} - \dfrac{1}{s}\right) = \dfrac{e^{-s}}{s-1} - \dfrac{e^{-2s}}{s-1} - \dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s}$

   $y(t) = L^{-1}\left[\dfrac{e^{-s}}{s-1} - \dfrac{e^{-2s}}{s-1} - \dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s}\right]$,

   $y(t) = H(t-1)e^{t-1} - H(t-2)e^{t-2} - H(t-1) + H(t-2) = H(t-1)(e^{t-1} - 1) + H(t-2)(-e^{t-2} + 1)$

$$y(t) = \begin{cases} 0, & \text{for } 0 \leq t < 1 \\ e^{t-1} - 1, & \text{for } 1 \leq t < 2 \\ e^{t-1} - e^{t-2}, & \text{for } t \geq 2 \end{cases}$$

3. Using the unit impulse response function and convolution find the solution to the initial-value problem

$$y'' + 9y = g(t), \qquad y(0) = 1, \quad y'(0) = 0,$$

where $g(t)$ is a piecewise continuous function.

Solution: First, we find the unit impulse response function for the equation.

$L[e'' + 9e] = (s^2 + 9)E(s), \quad E(s) = \dfrac{1}{s^2+9}, \quad e(t) = L^{-1}\left[\dfrac{1}{s^2+9}\right] = \dfrac{1}{3} \cdot L^{-1}\left[\dfrac{3}{s^2+3^2}\right] = \dfrac{1}{3}\sin 3t.$

Then the state-free solution is $y_s(t) = e{*}g\,(t) = \dfrac{1}{3}\displaystyle\int_0^t \sin(3(t-u))\,g(u)\,du \quad \left[= \dfrac{1}{3}\displaystyle\int_0^t \sin(3u)\,g(t-u)\,du\right]$

The imput-free solution is $y_i(t) = e'(t) + 0 \cdot e(t) = \cos 3t$

Therefore the solution is $y(t) = \cos 3t + \dfrac{1}{3}\displaystyle\int_0^t \sin(3(t-u))\,g(u)\,du$

4. For the initial-value problem $\quad y' = t(2y + t), \quad y(0) = 1$

calculate the second iteration y_2 of Euler's method with step size $h = 0.1$.

Solution: $t_0 = 0, \ y_0 = 1,$

$y_1 = y_0 + f(t_0, y_0)h = 1 + 0 = 1, \quad t_1 = t_0 + h = 0.1,$

$y_2 = y_1 + f(t_1, y_1)h = 1 + (0.1)(2 \cdot 1 + 0.1)(0.1) = 1 + 0.021 = 1.021$

5. Consider the initial value problem

$$y'' - 5y' + 2y = 2t^3, \qquad y(0) = 2, \quad y'(0) = 1$$

(a) Write the IVP as a system of first order equations.

Solution: $y'' = 5y' - 2y + 2t^3, \ x_1 = y, \ x_2 = y' = x_1', \ x_2' = y''.$

Then the system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 + 5x_2 + 2t^3 \\ x_1(0) &= 2 \\ x_2(0) &= 1 \end{aligned}$$

(b) Write the obtained system in vector form. Don't use matrices. Define all vectors.

Solution: Denote $\bar{\mathsf{x}}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \bar{\mathsf{x}}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \bar{\mathsf{f}} = \begin{pmatrix} f_1(t,\bar{\mathsf{x}}) \\ f_2(t,\bar{\mathsf{x}}) \end{pmatrix}$

where $f_1(t,\bar{\mathsf{x}}) = x_2 \quad f_2(t,\bar{\mathsf{x}}) = -2x_1 + 5x_2 + 2t^3.$

Then vector form is

$$\bar{\mathsf{x}}'(t) = \bar{\mathsf{f}}(t,\bar{\mathsf{x}}), \qquad \bar{\mathsf{x}}(0) = \bar{\mathsf{x}}_0$$

An alternative solution:

$$\bar{\mathsf{x}}'(t) = \begin{pmatrix} x_2 \\ -2x_1 + 5x_2 + 2t^3 \end{pmatrix}, \qquad \bar{\mathsf{x}}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

6. For the system of differential equations

$$x' = 4x - 2x^2 - xy$$
$$y' = 4y - xy - 2y^2$$

(a) find x-nullcline and y-nullcline.

Solution: x-nullcline: $4x - 2x^2 - xy = 0, \; x(4 - 2x - y) = 0.$

The x-nullcline is the union of two lines $x = 0$ and $y = -2x + 4$

y-nullcline: $4y - xy - 2y^2 = 0, \; y(4 - x - 2y) = 0.$

The y-nullcline is the union of two lines $y = 0$ and $y = -\frac{1}{2}x + 2$

(b) There are four equilibrium points.

bonus problem Find the inverse Laplace transform of the function $F(s) = \dfrac{s^2-4}{(s^2+4)^2}$

Solution: $L[t\cos 2t](s) = -F'(s)$, where $F(s) = L[\cos 2t](s) = \dfrac{s}{s^2+4}$

Check that $F'(s) = \dfrac{s^2 + 4 - s(2s)}{(s^2+4)^2} = \dfrac{4 - s^2}{(s^2+4)^2} = -\dfrac{s^2-4}{(s^2+4)^2}.$

Then $L[t\cos 2t](s) = -F'(s) = \dfrac{s^2-4}{(s^2+4)^2}.$

Hence $L^{-1}\left[\dfrac{s^2-4}{(s^2+4)^2}\right] = t\cos 2t$