Math 0290 **Midterm Exam 2**

Spring 2014 **S o l u t i o n s**

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1. Use the Laplace transform to solve the initial-value problem $y'' + 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = -5$. Show all the work.

   Solution: $L[y'' + 4y' + 4y] = L[0]$, $s^2Y - 3s + 5 + 4(sY - 3) + 4Y = 0$, $(s^2 + 4s + 4)Y = 3s + 7$,

   $$Y(s) = \frac{3s+7}{(s+2)^2} = \frac{3}{s+2} + \frac{1}{(s+2)^2}$$

   $$y(t) = L^{-1}\left[\frac{3}{s+2} + \frac{1}{(s+2)^2}\right], \quad y(t) = 3e^{-2t} + te^{-2t} = (3+t)e^{-2t}$$

2. Find the unit impulse response to the equation $e'' + 6e' + 10e = \delta(t)$.

   Solution: $L[e'' + 6e' + 10e] = 1$, $(s^2 + 6s + 10)E = 1$, $((s+3)^2 + 1)E = 1$

   $$E(s) = \frac{1}{(s+3)^2 + 1}$$

   $$e(t) = L^{-1}\left[\frac{1}{(s+3)^2 + 1}\right], \quad e(t) = e^{-3t}\sin t$$

3. Find the solution to the initial-value problem $y'' + 6y' + 10y = g(t)$, $y(0) = 0$, $y'(0) = 5$, where $g(t)$ is a piecewise continuous function.

   Solution: In the previous problem we found that the unit impulse response function for this equation is $e(t) = e^{-3t}\sin t$.

   Then the state-free solution is $y_s(t) = e * g\,(t) = \int_0^t e^{-3(t-u)}\sin(t-u)\,g(u)\,du$

   The imput-free solution is $y_i(t) = 0 \cdot e'(t) + 5e(t) = 5e^{-3t}\sin t$

   Therefore the solution is $y(t) = 5e^{-3t}\sin t + \int_0^t e^{-3(t-u)}\sin(t-u)\,g(u)\,du$

4. For the initial-value problem $y' = 3t - y$, $y(0) = 2$ calculate the first two iterations of Euler's method with step size $h = 0.2$.

   Solution: $t_0 = 0$, $y_0 = 2$, $y_1 = y_0 + f(t_0, y_0)h = 2 + (3 \cdot 0 - 2)(0.2) = 2 - 0.4 = 1.6$,

   $t_1 = t_0 + h = 0.2$, $y_2 = y_1 + f(t_1, y_1)h = 1.6 + (3 \cdot 0.2 - 1.6)(0.2) = 1.6 - 0.2 = 1.4$

5. For the system of differential equations

$x' = (y-x)(y+x^2-2)$

$y' = (y+x)(y-2)$

(a) find x-nullclines and y-nullclines.

Solution: $(y-x)(y+x^2-2)=0,\ \ y-x=0,\ \ y+x^2-2=0.$

Two x-nullclines: the line $y=x$ and the parabola $y=-x^2+2$

$(y+x)(y-2)=0,\ \ y+x=0,\ \ y-2=0.$

Two y-nullclines: lines $y=-x$ and $y=2$

(b) calculate the coordinates of the equilibrium points.

Solution: $(-1,1)$, $(0,0)$, $(2,-2)$, $(0,2)$, and $(2,2)$.

6. Find the general solution to the system $y' = Ay$, where $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.
Write the answer as a single vector.

Solution: $T=-2,\ \ D=0.$ Characteristic equation is $\lambda^2+2\lambda=0$.

Its roots are $\lambda_1=0$ and $\lambda_2=-2$.

$(A-\lambda_1 I)\bar{\mathsf{v}}_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \bar{\mathsf{v}}_1 = 0,\quad \bar{\mathsf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A-\lambda_2 I)\bar{\mathsf{v}}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \bar{\mathsf{v}}_2 = 0,\quad \bar{\mathsf{v}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The fundamental set of solutions is

$$\bar{\mathsf{y}}_1(t) = e^{0t}\,\bar{\mathsf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{\mathsf{y}}_2(t) = e^{-2t}\,\bar{\mathsf{v}}_2 = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

The general solution is $\bar{\mathsf{y}}(t) = C_1\bar{\mathsf{y}}_1(t) + C_2\bar{\mathsf{y}}_2(t) = \begin{pmatrix} C_1 + C_2 e^{-2t} \\ C_1 - C_2 e^{-2t} \end{pmatrix}$

bonus problem Find real valued solutions $y_1(t)$ and $y_2(t)$ of the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= -5y_1 + 3y_2 \end{aligned}$$

Solution: $A = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$ $T = 4,\ D = 13,\ \alpha = 2, \beta = 3,\ \lambda = 2 + 3i$, and $\bar{\lambda} = 2 - 3i$.

Eigenvectors are $\bar{\mathsf{v}}_1 = 2\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\bar{\mathsf{v}}_2 = 2\begin{pmatrix} 0 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

The general solution is $\bar{\mathsf{y}}(t) = C_1 e^{2t}(\cos 3t\,\bar{\mathsf{v}}_1 - \sin 3t\,\bar{\mathsf{v}}_2) + C_2 e^{2t}(\cos 3t\,\bar{\mathsf{v}}_2 + \sin 3t\,\bar{\mathsf{v}}_1) =$

$$C_1 e^{2t}\begin{pmatrix} 2\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 2\sin 3t \\ 3\cos 3t + \sin 3t \end{pmatrix} = e^{2t}\begin{pmatrix} 2C_1\cos 3t + 2C_2\sin 3t \\ (C_1 + 3C_2)\cos 3t + (-3C_1 + C_2)\sin 3t \end{pmatrix}$$

$$y_1(t) = 2e^{2t}(C_1\cos 3t + C_2\sin 3t) \quad y_2(t) = e^{2t}\left((C_1 + 3C_2)\cos 3t + (-3C_1 + C_2)\sin 3t\right)$$