Math 0290 **Midterm Exam 1**

Fall 2015 **S o l u t i o n s**

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1. (10 points) A 0.25 kg mass is attached to a spring having a spring constant 9 kg/s^2. The system is displaced 0.3 m from its equilibrium position and released from rest. If there is no dumping present, find the amplitude, frequency, and phase angle of the resulting motion.

   Solution: The model is described by the IVP: $0.25x'' + 9x = 0, \quad x(0) = 0.3, \; x'(0) = 0.$

   After multiplication the equation by 4 we get $x'' + 36x = 0.$

   The natural frequency is $\omega_0 = \sqrt{36} = 6$ and the general solution is $x(t) = c_1 \cos 6t + c_2 \sin 6t.$

   Then $x'(t) = -6c_1 \sin 6t + 6c_2 \cos 6t$

   The initial conditions give $x(0) = c_1 = 0.3, \; x'(0) = 6c_2 = 0, \; c_2 = 0.$

   Hence $x(t) = 0.3 \cos 6t$

   The amplitude is 0.3, frequency is 6, and phase angle is 0.

   (The angle is 6 rad/sec or $6/2\pi = 3/\pi$ Hz).

2. Solve the initial-value problem. Find the interval of existence of the solution. Show all the work. Mention type of the given differential equation.

   (a) (15 points) $y \ln y + x\, y' = 0, \qquad y(1) = e^{-2}, \quad$ where $y' = \dfrac{dy}{dx}.$

   Solution: It is a **separable** equation

   $$x\frac{dy}{dx} = -y \ln y, \quad \frac{dy}{y \ln y} = -\frac{dx}{x}, \quad \int \frac{dy}{y \ln y} = -\int \frac{dx}{x},$$

   Substitution: $u = \ln y, \; du = \dfrac{dy}{y},$

   $\ln|\ln y| = -\ln|x| + C = \ln|x^{-1}| + C, \quad \ln y = Ax^{-1} = A/x, \quad y = e^{A/x}.$

   $y(1) = e^A = e^{-2}, \quad A = -2, \quad y(x) = e^{-2/x}.$

   $x \neq 0,$ the interval of existence is $(0, \infty).$

(b) (15 points) $(x^2+1)y' + 2xy = 6x, \qquad y(0) = -1.$

Solution: Divide both sides by x^2+1 to get a first order **linear** differential equation:
$y' + \dfrac{2x}{x^2+1}\,y = \dfrac{6x}{x^2+1}.$

The integrating factor is $u = e^{\int \frac{2x}{x^2+1}\,dx} = e^{\ln(x^2+1)} = x^2+1$. Then

$(x^2+1)\left(y' + \dfrac{2x}{x^2+1}y\right) = 6x, \quad (x^2+1)y' + 2xy = 6x, \quad \big((x^2+1)y\big)' = 6x,$

$(x^2+1)y = \int 6x\,dx = 3x^2 + C, \quad y = \dfrac{3x^2+C}{x^2+1}.$

$y(0) = C = -1, \quad y(x) = \dfrac{3x^2-1}{x^2+1}.$ The interval of existence is $(-\infty, \infty)$.

You can solve this equation as separable: $\dfrac{dy}{dx} = \dfrac{2x(3-y)}{x^2+1}, \int \dfrac{dy}{3-y} = \int \dfrac{2x}{x^2+1}\,dx,$

$-\ln|y-3| = \ln(x^2+1) + C, \; y = 3 + \dfrac{A}{x^2+1}, \; A = -4, \; y = 3 - \dfrac{4}{x^2+1}.$

3. Consider the equation $y'' + 3y' - 4y = 0.$

(a) (10 points) Show that $y_1 = e^t$ and $y_2 = e^{-4t}$ are solutions of the given equation.

Solution: $y_1'' = y_1' = e^t$, $y_1'' + 3y_1' - 4y_1 = e^t + 3e^t - 4e^t = 0$. Hence y_1 is a solution.
$y_2' = -4e^{-4t}$, $y_2'' = 16e^{-4t}$, $y_2''+3y_2'-4y_2 = 16e^{-4t}-12e^{-4t}-4e^{-4t} = 0$. Hence y_2 is a solution.

(b) (10 points) Use Wronskian to show that y_1 and y_2 are linearly independent and hence form the fundamental set of solutions.

Solution: The Wronskian is
$W = y_1y_2' - y_1'y_2 = e^t \cdot \left(-4e^{-4t}\right) - e^te^{-4t} = -4e^{-3t} - e^{-3t} = -5e^{-3t}.$
$W(0) = -5 \neq 0$. Hence y_1 and y_2 are linearly independent.
[Note: There is no a statement that says if $W(t) \neq 0$ then y_1 and y_2 are LI].

(c) (10 points) Find the solution of the given differential equation satisfying the initial conditions $y(0) = 3, \; y'(0) = -2.$

Solution: The general solution is $y(t) = c_1y_1 + c_2y_2 = c_1e^t + c_2e^{-4t}, \; y'(t) = c_1e^t - 4c_2e^{-4t}$
Then $y(0) = c_1 + c_2 = 3$ and $y'(0) = c_1 - 4c_2 = -2;$ $c_1 = 2$ and $c_2 = 1$.
Therefore $y(t) = 2e^t + e^{-4t}$.

4. In the previous problems it was proven that $y_1 = e^t$, $y_2 = e^{-4t}$ form the fundamental set of solutions of the homogeneous equation $y'' + 3y' - 4y = 0$. Use this result to find a general solution of the equation

$$y'' + 3y' - 4y = 10e^t$$

(a) (15 points) by using the method of undetermined coefficients.

Solution: The right hand side of the given non-homogeneous DE coincides with y_1. Therefore to find a particular solution we use the form $y_p = ate^t$.

Then $y_p' = ae^t + ate^t$, $y_p'' = 2ae^t + ate^t$

$y_p'' + 3y_p' - 4y_p = 2ae^t + ate^t + 3ae^t + 3ate^t - 4ate^t = 5ae^t = 10e^t$, $a = 2$, $y_p = 2te^t$.

The general solution is $y(t) = C_1e^t + C_2e^{-4t} + 2te^t$.

(b) (15 points) by using the method of variation of parameters. You may use the Wronskian found before and formulas for evaluation v_1 and v_2 directly.

Solution: $W = -5e^{-3t}$. $v_1 = -\int \frac{y_2 \cdot 10e^t}{W}\,dt = -\int \frac{e^{-4t} \cdot 10e^t}{-5e^{-3t}}\,dt = 2\int dt = 2t$

$v_2 = \int \frac{y_1 \cdot 10e^t}{W}\,dt = \int \frac{e^t \cdot 10e^t}{-5e^{-3t}}\,dt = -2\int e^{5t}\,dt = -\frac{2}{5}e^{5t}$

Hence $y_p = 2te^t - \frac{2}{5}e^{5t}e^{-4t} = 2te^t - \frac{2}{5}e^t$.

The general solution is $y(t) = C_0e^t + C_2e^{-4t} + 2te^t - \frac{2}{5}e^t = C_1e^t + C_2e^{-4t} + 2te^t$,

where $C_1 = C_0 - \frac{2}{5}$.

bonus problem (15 points extra) Find the general solution of the equation $ty'' = y'$.

Solution: Substitution: $x = y'$. Then $y'' = x'$.

So, the given equation becomes $tx' = x$,

$\int \frac{dx}{x} = \int \frac{dt}{t}$, $x = at$, $y' = at$. Then $y = \frac{at^2}{2} + c$ or $y = bt^2 + c$.