Math 0290 **Midterm Exam 2**

Fall 2015 **S o l u t i o n s**

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1. Find the inverse Laplace transform of the function $F(s) = \dfrac{6}{s^2 - 3s}$, $s \geq 3$. Show all your work.

Solution: $\dfrac{6}{s^2 - 3s} = \dfrac{6}{s(s-3)} = \dfrac{2}{s-3} - \dfrac{2}{s}$

$y(t) = L^{-1}\left[\dfrac{2}{s-3} - \dfrac{2}{s}\right] = 2L^{-1}\left[\dfrac{1}{s-3}\right] - 2L^{-1}\left[\dfrac{1}{s}\right]$, $y(t) = 2e^{3t} - 2$

2. By using Laplace transform solve the initial-value problem $y' - 3y = g(t)$, $y(0) = 0$, where

$$g(t) = \begin{cases} 0, & 0 \leq t < 4 \\ 6, & t \geq 4 \end{cases}$$

Create a piecewise definition for your solution that does not use the Heaviside function.

Show all your work. You may use results from the previous problem.

Solution: $g(t) = 0 \cdot [H(t) - H(t-4)] + 6 \cdot H(t-4) = 6H(t-4)$,

$L[y' - 3y] = (s-3)Y(s)$, $L[g(t)] = L\left[6H(t-4)\right] = \dfrac{6e^{-4s}}{s}$

$(s-3)Y(s) = \dfrac{6e^{-4s}}{s}$,

$Y(s) = e^{-4s} \cdot \dfrac{6}{s(s-3)} = e^{-4s}\left(\dfrac{2}{s-3} - \dfrac{2}{s}\right) = \dfrac{2e^{-4s}}{s-3} - \dfrac{2e^{-4s}}{s}$

$y(t) = L^{-1}\left[\dfrac{2e^{-4s}}{s-3} - \dfrac{2e^{-4s}}{s}\right] = 2L^{-1}\left[\dfrac{e^{-4s}}{s-3}\right] - 2L^{-1}\left[\dfrac{e^{-4s}}{s}\right]$,

$y(t) = 2H(t-4)e^{3(t-4)} - 2H(t-4) = H(t-4)(2e^{3t-12} - 2)$

$$y(t) = \begin{cases} 0, & 0 \leq t < 4 \\ 2e^{3t-12} - 2, & t \geq 4 \end{cases}$$

3. Using the unit impulse response function and convolution find the solution to the initial-value problem

$$y'' + 25y = g(t), \qquad y(0) = 1, \quad y'(0) = 5,$$

where $g(t)$ is a piecewise continuous function.

Solution: First, we find the unit impulse response function for the equation.

$L[e'' + 25e] = (s^2 + 25)E(s), \quad E(s) = \dfrac{1}{s^2 + 25},$

$e(t) = L^{-1}\left[\dfrac{1}{s^2 + 25}\right] = \dfrac{1}{5} \cdot L^{-1}\left[\dfrac{5}{s^2 + 5^2}\right] = \dfrac{1}{5}\sin 5t.$

Then the state-free solution is $y_s(t) = e{*}g\,(t) = \dfrac{1}{5}\displaystyle\int_0^t \sin(5(t-u))\,g(u)\,du \quad \left[= \dfrac{1}{5}\displaystyle\int_0^t \sin(5u)\,g(t-u)\,du\right]$

The imput-free solution is $y_i(t) = e'(t) + 5e(t) = \cos 5t + \sin 5t$

Therefore the solution is $y(t) = \cos 5t + \sin 5t + \dfrac{1}{5}\displaystyle\int_0^t \sin(5(t-u))\,g(u)\,du$

4. For the initial-value problem $y' = 3t(y + t), \quad y(0) = 2$

calculate the second iteration y_2 of Euler's method with step size $h = 0.1$.

Solution: $t_0 = 0, \; y_0 = 2,$

EM: $y_{n+1} = y_n + f(t_n, y_n)h = y_n + 0.3t_n(y_n + t_n),$

$y_1 = y_0 + 0.3t_0(y_0 + t_0) = 2 + 0 = 2, \quad t_1 = t_0 + h = 0.1,$

$y_2 = y_1 + 0.3t_1(y_1 + t_1) = 2 + (0.3)(0.1)(2.1) = 2 + 0.063 = 2.063$

5. For the system of differential equations

$$x' = x - xy$$
$$y' = x^2y - y^2$$

(a) find x-nullcline and y-nullcline. Draw a plot.

Solution: x-nullcline: $x - xy = 0, \; x(1 - y) = 0.$

The x-nullcline is the union of two lines $x = 0$ and $y = 1$

y-nullcline: $x^2y - y^2 = 0, \; y(x^2 - y) = 0.$

The y-nullcline is the union of two lines $y = 0$ and $y = x^2$

(b) find equilibrium points. Mark them on the plot.

Solution: There are three equilibrium points $(0,0)$, $(1,1)$, and $(-1,1)$.

6. Find the general solution to the system $\bar{y}' = A\bar{y}$, where $A = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}$.
Write the answer as a single vector.

Solution: $T = 7$, $D = -8 + 18 = 10$. Characteristic equation is $\lambda^2 - 7\lambda + 10 = 0$.

Its roots are $\lambda_1 = 2$ and $\lambda_2 = 5$.

$(A - \lambda_1 I)\bar{v}_1 = \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix} \bar{v}_1 = 0, \quad \bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(A - \lambda_2 I)\bar{v}_2 = \begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix} \bar{v}_2 = 0, \quad \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The fundamental set of solutions is

$$\bar{y}_1(t) = e^{2t}\,\bar{v}_1 = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}$$

$$\bar{y}_2(t) = e^{5t}\,\bar{v}_2 = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}$$

The general solution is $\bar{y}(t) = C_1\bar{y}_1(t) + C_2\bar{y}_2(t) = \begin{pmatrix} 2C_1e^{2t} + C_2e^{5t} \\ C_1e^{2t} + C_2e^{5t} \end{pmatrix}$

bonus problem Find the inverse Laplace transform of the function $F(s) = \dfrac{s^2}{(s^2+1)^2}$

Solution: $L^{-1}\left[\dfrac{s^2}{(s^2+1)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+1} \cdot \dfrac{s}{s^2+1}\right] = \cos t * \cos t = \int\limits_0^t \cos u \cos(t-u)\,du$

$= \int\limits_0^t \cos u\,(\cos t \cos u + \sin t \sin u)\;du = \cos t \int\limits_0^t \frac{1}{2}\,(\cos 2u + 1)\;du + \sin t \int\limits_0^t \frac{1}{2}\sin 2u\,du$

$= \cos t\left[\frac{1}{4}\sin 2u + \frac{1}{2}u\right]_0^t + \sin t\left[-\frac{1}{4}\cos 2u\right]_0^t = \cos t\left[\frac{1}{4}\sin 2t + \frac{1}{2}t\right] + \sin t\left[-\frac{1}{4}\cos 2t + \frac{1}{4}\right]$

$= \frac{1}{4}\cos t \sin 2t - \frac{1}{4}\sin t \cos 2t + \frac{1}{2}t\cos t + \frac{1}{4}\sin t = \frac{1}{4}\sin t + \frac{1}{2}t\cos t + \frac{1}{4}\sin t$

$= \frac{1}{2}\sin t + \frac{1}{2}t\cos t$