Math 0290 **Midterm Exam 1**

Spring 2016 **S o l u t i o n s**

---

1. (a) (15 points) Solve the initial value problem $x' = 1 + x\,\tan t, \qquad x(0) = 5$

   Solution: $x' - (\tan t)\,x = 1$. It is a **first order linear** equation.

   The integrating factor is $u = e^{-\int \tan t\,dt} = e^{\ln(\cos t)} = \cos t$.

   (Note: $-\int \tan t\,dt = -\int \frac{\sin t}{\cos t}\,dt$ and use the substitution $v = \cos t$ to evaluate the integral).

   Then $(\cos t)\,x' - (\sin t)\,x = \cos t, \quad ((\cos t)\,x)' = \cos t, \quad (\cos t)\,x = \sin t + c$

   So, the general solution is $x(t) = \tan t + c\sec t$.

   IC: $x(0) = c = 5$.

   Therefore, $x(t) = \tan t + 5\sec t$.

   (b) (5 points) Find the interval of existence.

   Solution: The solution is undefined for all t where $\cos t = 0$. That means $t = \frac{\pi}{2} + \pi n$, where n is an integer number.

   Because the initial condition is defined at $t = 0$ the interval of existence is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

2. (15 points) Solve the initial value problem $y' = \dfrac{\sin x}{2y}, \qquad y(\pi) = -2.$

   Solution: It is a **separable** equation.

   $\dfrac{dy}{dx} = \dfrac{\sin x}{2y}, \quad 2y\,dy = \sin x\,dx, \quad \int e^y dy = \int 2x\,dx, \quad \int 2y\,dy = \int \sin x\,dx, \quad y^2 = -\cos x + C.$

   Then general solution has two branches $y(x) = -\sqrt{-\cos x + C}$ and $y(x) = \sqrt{-\cos x + C}$

   The initial condition $y(\pi) = -2$ says that for a particular solution we have to choose the negative radical. Then

   $y(\pi) = -\sqrt{-\cos\pi + C} = -2, \ \sqrt{1 + C} = 2, \ C = 3.$

   Therefore, the particular solution is $y(x) = -\sqrt{-\cos x + 3}$.

3. (15 points) Consider a simple RLC circuit with $R = 2\ \Omega$, $C = \frac{1}{4}$ F, $E = 12\,\text{V}$ and there is no inductor. Find $I(t)$ if $I(0) = 2\,\text{A}$.

Solution: We have the IVP $2\,\dfrac{dQ}{dt} + 4Q = 12, \quad I(0) = 2.$

It is a first order linear equation $\dfrac{dQ}{dt} + 2Q = 6$. The integrating factor is $u(t) = e^{\int 2\,dt} = e^{2t}$

Then $\left(e^{2t}\,Q\right)' = 6\,e^{2t} \;\Rightarrow\; e^{2t}\,Q = 3\,e^{2t} + C \;\Rightarrow\; Q(t) = 3 + Ce^{-2t}$

[An alternative solution:

$\dfrac{dQ}{dt} = -2(Q-3),\ \int \dfrac{dQ}{Q-3} = -2\int dt,\ \ln|Q-3| = -2t + c,\ Q - 3 = Ce^{-2t},\ Q = 3 + Ce^{-2t}$]

Then $I(t) = \dfrac{dQ}{dt} = -2Ce^{-2t}$

$I(0) = -2C = 2 \;\Rightarrow\; C = -1. \qquad I(t) = 2e^{-2t}.$

4. (15 points) For the initial-value problem

$$y' = \frac{y}{t}, \qquad y(1) = 2$$

find an approximate value of $y(1.2)$. Use Euler's method with the step size $h = 0.1$. Simplify your answer.

Solution: Euler's Method: $y_{n+1} = y_n + f(t_n, y_n)h = y_n + \dfrac{y_n}{t_n}(0.1), \quad y_{n+1} = y_n\left(1 + \dfrac{0.1}{t_n}\right)$

$y_0 = 2,\ t_0 = 1, \quad y(1.1) \approx y_1 = 2 + \dfrac{y_0}{t_0}\cdot 0.1 = 2 + \dfrac{2}{1}\cdot 0.1 = 2.2$

$t_1 = t_0 + h = 1.1, \quad y(1.2) \approx y_2 = 2 + \dfrac{y_1}{t_1}\cdot 0.1 = 2 + \dfrac{2.2}{1.1}\cdot 0.1 = 2.2 + 0.2 = 2.4.$

5. A 2-kg mass when attached to a spring, stretches the spring to a distance of 4.9 m.

(a) (5 points) Calculate the spring constant.

Solution: $k = \dfrac{mg}{x_0} = \dfrac{2\text{ kg} \times 9.8\text{ m/s}^2}{4.9\text{ m}} = 4\text{ kg/s}^2.$

(b) (10 points) The system is placed in a viscous medium that supplies a damping constant $\mu = 6$ kg/s. The system is allowed to come to rest. Then the mass is displaced 2 m in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 m/s in the downward direction. Find the position of the mass as a function of time.

Solution: IVP: $2y'' + 6y' + 4y = 0, \quad y(0) = 2,\ y'(0) = 1$

Characteristic equation: $2\lambda^2 + 6\lambda + 4 = 0, \quad \lambda^2 + 3\lambda + 2 = 0, \quad (\lambda+2)(\lambda+1) = 0$

$\lambda_1 = -2,\ \lambda_2 = -1 \;\Rightarrow\; y(t) = c_1e^{-2t} + c_2e^{-t}$ is a general solution.

$y'(t) = -2c_1e^{-2t} - c_2e^{-t}$

ICs: $y(0) = c_1 + c_2 = 2,\ y'(0) = -2c_1 - c_2 = 1 \quad \Rightarrow \quad c_1 = -3,\ c_2 = 5.$

$y(t) = -3e^{-2t} + 5e^{-t}$

6. For the equation $\quad y'' - 4y' - 5y = 12e^{-t}.$

(a) (5 points) Find the fundamental set of solutions of the corresponding homogeneous equation.

Solution: Homogeneous equation is $y'' - 4y' - 5y = 0.$

Char. eq. is $\lambda^2 - 4\lambda - 5 = 0.$ It has distinct real roots $\lambda_1 = -1$ and $\lambda_2 = 5$.

FSS: $y_1(t) = e^{-t},\ y_2(t) = e^{5t}.$

(b) (10 points) Find a particular solution by using the method of undetermined coefficients.

Solution: Since the function e^{-t} in the forced term is in the FSS we are looking for a trial solution in the form $y_p(t) = ate^{-t}.$

Then $y_p' = a(1-t)e^{-t}$ and $y_p'' = a(-2+t)e^{-t}.$

$y_p'' - 4y_p' - 5y_p = a(-2+t-4+4t-5t)e^{-t} = -6ae^{-t} = 12e^{-t}. \quad \Rightarrow \quad a = -2.$

Hence $y_p(t) = -2te^{-t}.$

(c) (5 points) Find the general solution.

Solution: $y(t) = c_1y_1 + c_2y_2 + y_p = c_1e^{-t} + c_2e^{5t} - 2te^{-t} = (c_1 - 2t)\,e^{-t} + c_2e^{5t}$

bonus problem (15 points extra) Suppose $y(x)$ is a differentiable nonnegative function with $y(0) = 0.$ Find $y(x)$ if the area under the curve $y = y(x)$ from 0 to x $(x > 0)$ is always equal to one fourth the area of the rectangle with vertices at $(0, 0)$ and $(x, y(x)).$

Solution: We have $\int_0^x y(t)\,dt = \frac{1}{4}xy.$

After differentiating both sides with respect to x we get $y = \frac{1}{4}(y + xy')$ or $xy' = 3y.$

It is a separable equation $\dfrac{dy}{y} = \dfrac{3}{x}\,dx, \quad \ln y = \ln x^3 + C.$

$y(x) = ax^3,$ where a is any positive constant.