Math 0290 **Midterm Exam 2**

Fall 2016 **S o l u t i o n s**

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1. By using Laplace transform solve the initial-value problem

$$y'' + 4y = 4\cos 2t, \quad y(0) = 0, \quad y'(0) = 0$$

Do not use the convolution.

Solution: After applying Laplace transform to the equation we get $(s^2+4)Y(s) = \dfrac{4s}{s^2+4}$

Then $Y(s) = \dfrac{4s}{(s^2+4)^2} = -\left(\dfrac{2}{s^2+4}\right)' = -\left(L[\sin 2t]\right)' = L[t\sin 2t]$

Therefore $y(t) = t\sin 2t$.

2. Consider the initial-value problem $y' + y = g(t)$, $y(0) = 0$, where

$$g(t) = \begin{cases} 0, & \text{for } \; 0 \le t < 3 \\ t, & \text{for } \; t \ge 3 \end{cases}$$

(a) Describe the function $g(t)$ in terms of the Heaviside function.

Solution: $g(t) = tH(t-3) = (t-3)H(t-3) + 3H(t-3)$

(b) By using Laplace transform solve the initial-value problem. Do not use the convolution. Hint:

$$\frac{3s+1}{s^2(s+1)} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}$$

Solution: $L[y' + y] = (s+1)Y(s)$

$L[g(t)] = L\left[(t-3)H(t-3) + 3H(t-3)\right] = e^{-3s}\cdot\dfrac{1}{s^2} + 3e^{-3s}\cdot\dfrac{1}{s} = e^{-3s}\cdot\dfrac{3s+1}{s^2}$

Then $(s+1)Y(s) = e^{-3s}\cdot\dfrac{3s+1}{s^2}$

$Y(s) = e^{-3s}\cdot\dfrac{3s+1}{s^2(s+1)} = e^{-3s}\left(\dfrac{2}{s} + \dfrac{1}{s^2} - \dfrac{2}{s+1}\right) = \dfrac{2e^{-3s}}{s} + \dfrac{e^{-3s}}{s^2} - \dfrac{2e^{-3s}}{s+1}$

$$y(t) = L^{-1}\left[\frac{2e^{-3s}}{s} + \frac{e^{-3s}}{s^2} - \frac{2e^{-3s}}{s+1}\right] = 2L^{-1}\left[\frac{e^{-3s}}{s}\right] + L^{-1}\left[\frac{e^{-3s}}{s^2}\right] - 2L^{-1}\left[\frac{e^{-3s}}{s+1}\right]$$

$$y(t) = 2H(t-3) + (t-3)H(t-3) - 2e^{-(t-3)}H(t-3) = H(t-3)(t-1-2e^{3-t})$$

(c) Create a piecewise definition for your solution $y(t)$ that does not contain the Heaviside function.

Solution:

$$y(t) = \begin{cases} 0, & 0 \le t < 3 \\ t - 1 - 2e^{3-t}, & t \ge 4 \end{cases}$$

3. Use Laplace transform and the convolution to find a solution to the initial-value problem

$$y'' + 4y = 2\sqrt{t}, \qquad y(0) = 1, \quad y'(0) = -6$$

Solution: First, we find the unit impulse response function for the equation.

$$e(t) = L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2} \cdot L^{-1}\left[\frac{2}{s^2+2^2}\right] = \tfrac{1}{2}\sin 2t.$$

Then the state-free solution is $y_s(t) = \left(e \, * \, 2\sqrt{t}\right)(t) = \int_0^t \sin(2(t-u))\sqrt{u}\, du \quad \left[= \int_0^t \sin(2u)\sqrt{t-u}\, du\right]$

The imput-free solution is $y_i(t) = e'(t) - 6e(t) = \cos 2t - 3\sin 2t$

Therefore the solution is $y(t) = \cos 2t - 3\sin 2t + \int_0^t \sin(2(t-u))\sqrt{u}\, du$

4. For the homogeneous linear system

$$\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= -y_1 + 4y_2 \end{aligned}$$

(a) Find the corresponding 2×2 matrix A

Solution: The vector form of the system is $\bar{y}' = A\bar{y}$,

where $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

(b) Find eigenvalues of the matrix A

Solution: $T = 5, \;\; D = 4 + 2 = 6.$

Characteristic equation $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$
has two real roots $\lambda_1 = 2$ and $\lambda_2 = 3$.

(c) Find eigenvectors of the matrix A

Solution: Let $\bar{v}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be an eigenvector associated with the eigenvalue $\lambda_1 = 2$.

Then $(A - \lambda_1 I_2)\,\bar{v}_1 = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We get $-u_1 + 2u_2 = 0 \quad \Leftrightarrow \quad u_1 = 2u_2$. Take $u_2 = 1$ then $u_1 = 2$ and $\bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Let $\bar{v}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be an eigenvector associated with the eigenvalue $\lambda_2 = 3$.

Then $(A - \lambda_2 I_2)\,\bar{v}_2 = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We get $-u_1 + u_2 = 0 \quad \Leftrightarrow \quad u_1 = u_2$. Take $u_2 = 1$ then $u_1 = 1$ and $\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) Find the general solution of the system

Solution: The general solution is

$$\bar{y}(t) = C_1\, e^{2t}\,\bar{v}_1 + C_2\, e^{3t}\,\bar{v}_2 = \begin{pmatrix} 2C_1\, e^{2t} \\ C_1\, e^{2t} \end{pmatrix} + \begin{pmatrix} C_2\, e^{3t} \\ C_2\, e^{3t} \end{pmatrix} = \begin{pmatrix} 2C_1\, e^{2t} + C_2\, e^{3t} \\ C_1\, e^{2t} + C_2\, e^{3t} \end{pmatrix}$$

(e) Find $y_1(t)$ and $y_2(t)$.

Solution: $y_1(t) = 2C_1\, e^{2t} + C_2\, e^{3t}, \quad y_2(t) = C_1\, e^{2t} + C_2\, e^{3t}$

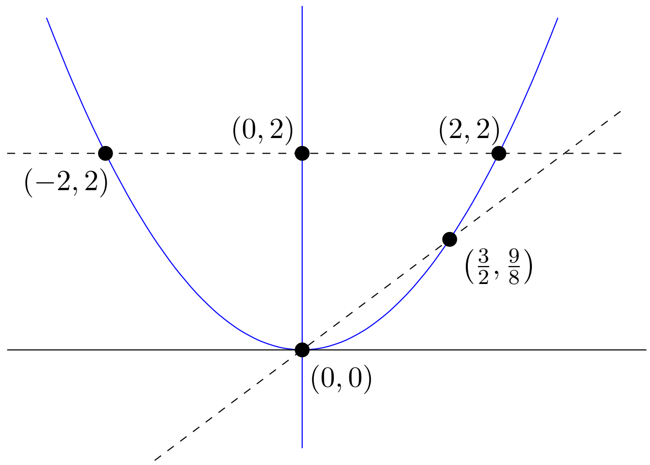
5. For the system of differential equations

$$x' = 2xy - x^3$$
$$y' = (y - 2)(4y - 3x)$$

(a) find x-nullcline and y-nullcline. Draw a plot.

Solution: x-nullcline: $2xy - x^3 = 0 \quad \Leftrightarrow \quad x(2y - x^2) = 0 \quad \Leftrightarrow \quad x = 0$ or $y = \frac{1}{2}x^2$.
Therefore, x-nullcline is a union of two curves, the line $x = 0$ and the parabola $y = \frac{1}{2}x^2$ (solid blue lines on the plot below).

y-nullcline: $(y - 2)(4y - 3x) = 0 \quad \Leftrightarrow \quad y = 2$ or $y = \frac{3}{4}x$.
Therefore, y-nullcline is a union of two lines $y = 2$ and $y = \frac{3}{4}x$ (dashed lines).



(b) find equilibrium points. Clearly mark them on the plot.

Solution: There are five equilibrium points $(0,0)$, $(-2,2)$,$(0,2)$, $(2,2)$, and $\left(\frac{3}{2}, \frac{9}{8}\right)$.

bonus problem Find the inverse Laplace transform of the function $G(s) = \dfrac{s-3}{(s^2-6s+10)^2}$

Solution: 1. $G(s) = -\frac{1}{2}\left(\dfrac{1}{s^2-6s+10}\right)'$. Indeed

$$\left(\frac{1}{s^2-6s+10}\right)' = \left((s^2-6s+10)^{-1}\right)' = -(s^2-6s+10)^{-2}(2s-6) = -2\frac{s-3}{(s^2-6s+10)^2} = -2G(s)$$

2. $\dfrac{1}{s^2-6s+10} = \dfrac{1}{s^2-6s+9+1} = \dfrac{1}{(s-3)^2+1} = L[e^{3t}\sin t]$.

3. Then, $L[t\,e^{3t}\,\sin t](s) = -\left(\dfrac{1}{s^2-6s+10}\right)' = 2G(s)$

4. Therefore, $G(s) = \frac{1}{2}\,L[t\,e^{3t}\,\sin t](s) = L[\frac{1}{2}\,t\,e^{3t}\,\sin t](s)$ and $L^{-1}\left[G(s)\right] = \frac{1}{2}\,t\,e^{3t}\,\sin t$