Math 0290 **Midterm Exam 1**

Spring 2016 **S o l u t i o n s**

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1. (15 points) A hospital received 200 mg of the isotope Iodine 131. After 14 days only 60 mg remained. Find the half-life of the isotope. Write answer in exact form.

   Solution: Let $N(t)$ be the number of remaining nuclei after time t. It satisfies the IVP $N' = -\lambda N,\ N(0) = 200$. Its solution is $N(t) = 200e^{-\lambda t}$

   To find λ (or $e^{-\lambda}$) we use the condition $N(14) = 60$. Then $200e^{-\lambda \cdot 14} = 60,\ \left(e^{-\lambda}\right)^{14} = .3$, $e^{-\lambda} = (.3)^{1/14}$

   So, we get $N(t) = 200(.3)^{t/14}\ \left(= 200e^{t\ln(0.3)/14}\right)$

   Let T be the half-life. Then $N(T) = 100$ or $200(.3)^{T/14} = 100,\ (.3)^{T/14} = .5,\ \frac{T}{14}\ln(.3) = \ln(.5)$

   Hence $T = 14\dfrac{\ln(.5)}{\ln(.3)}$ days.

2. (15 points) Determine a type of the given differential equation and find the solution of the initial value problem.

   $(3+t)x' + x = \sin t, \qquad x(0) = 0.$

   Solution: Divide both sides by $3+t$ to get a first order **linear** differential equation:
   $x' + (3+t)^{-1}\,x = (3+t)^{-1}\,\sin t.$

   The integrating factor is $u = e^{\int (3+t)^{-1}\,dt} = e^{\ln(3+t)} = 3+t$. Then

   $(3+t)x' + x = \sin t$ (which is the original equation), $\quad ((3+t)x)' = \sin t, \quad (3+t)x = -\cos t + c$

   So, the general solution is $x(t) = \dfrac{c - \cos t}{3+t}$

   The initial condition gives $x(0) = \dfrac{c-1}{3} = 0,\ c = 1.$

   Therefore, $x(t) = \dfrac{1 - \cos t}{3+t}.$

3. (15 points) Determine a type of the given differential equation and find its general solution.

   $y' = 2xe^{-y}, \quad$ where $y' = \dfrac{dy}{dx}.$

   Solution: It is a **separable** equation.

$e^y dy = 2x dx, \quad \int e^y dy = \int 2x\, dx, \quad e^y = x^2 + C.$

The general solution is $\quad y(x) = \ln(x^2 + C)$

Note also, that $\; y(x) = \ln|x^2 + C| \;$ is not a correct solution.

4. (15 points) A 0.2 kg mass is attached to a spring having a spring constant 5 kg/s^2. The system is displaced 0.3 m from its equilibrium position and released from rest. If there is no dumping present, find the amplitude, frequency, and phase angle of the resulting motion.

Solution: The model is described by the IVP: $\quad 0.2x'' + 5x = 0, \quad x(0) = 0.3, \; x'(0) = 0.$

After multiplication the equation by 5 we get $\quad x'' + 25x = 0.$

The natural frequency is $\omega_0 = \sqrt{25} = 5$ and the general solution is $x(t) = c_1 \cos 5t + c_2 \sin 5t.$

Then $x'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t$

The initial conditions give $\quad x(0) = c_1 = 0.3, \; x'(0) = 5c_2 = 0, \; c_2 = 0.$

Hence $\; x(t) = 0.3 \cos 5t$

The amplitude is 0.3, frequency is 5, and phase angle is 0.

(The frequency is 5 rad/sec or $5/2\pi = 2.5/\pi$ Hz).

5. Consider the equation $\quad y'' - 6y' + 9y = e^{3t}.$

(a) (10 points) Find the fundamental set of solutions of the corresponding homogeneous equation.

Solution: Homogeneous equation is $y'' - 6y' + 9y = 0.$

Char. eq. is $\lambda^2 - 6\lambda + 9 = 0.$ It has a repeated root $\lambda = 3.$

FSS: $y_1(t) = e^{3t}, \; y_2(t) = te^{3t}.$

(b) (10 points) Find a particular solution by using the method of variation of parameters.

Solution: $\quad W(t) = e^{3t} \cdot (1 + 3t)e^{3t} - 3e^{3t} \cdot te^{3t} = e^{6t}$

$$v_1 = -\int \frac{y_2 \cdot e^{3t}}{W}\, dt = -\int \frac{te^{6t}}{e^{6t}}\, dt = -\int t\, dt = -\frac{t^2}{2}$$

$$v_2 = \int \frac{y_1 \cdot e^{3t}}{W}\, dt = \int \frac{e^{3t} \cdot e^{3t}}{e^{6t}}\, dt = \int dt = t$$

Hence $\; y_p(t) = -\dfrac{t^2}{2} e^{3t} + t^2 e^{3t} = \frac{1}{2} t^2 e^{3t}.$

(c) (10 points) Find a particular solution by using the method of undetermined coefficients.

Solution: Since both e^{3t} and te^{3t} form FSS a trial solution has to be found in the form $y_p(t) = at^2e^{3t}$

Then $y_p' = (2at + 3at^2)e^{3t}$ and $y_p'' = (2a + 12at + 9at^2)e^{3t}$.

$y_p'' - 6y_p' + 9y_p = (2a + 12at + 9at^2 - 12at - 18at^2 + 9at^2)e^{3t} = 2ae^{3t} = e^{3t}$. So, $a = \frac{1}{2}$.

Hence $y_p(t) = \frac{1}{2}t^2e^{3t}$.

(d) (10 points) Find the general solution.

Solution: $y(t) = c_1y_1 + c_2y_2 + y_p = (c_1 + c_2t)e^{3t} + \frac{1}{2}t^2e^{3t} = \left(c_1 + c_2t + \frac{1}{2}t^2\right)e^{3t}$

bonus problem (15 points extra) Find the general solution of the equation $\dfrac{x'}{x} = \dfrac{1}{x} + \tan t$.

Solution: After multiplication both sides by x we get a linear first order equation: $x' - \tan t \cdot x = 1$.

Integrating factor is $u = \cos t$. Hint: $\displaystyle\int \tan t\, dt = \int \frac{\sin t}{\cos t}\, dt$ and use the substitution $u = \cos t$.

Then $(\cos t \cdot x)' = \cos t$, $\cos t \cdot x = \sin t + c$, $x(t) = \tan t + c \sec t$.