Math 0290 **Midterm Exam 2**

Spring 2016 **S o l u t i o n s**

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1. By using Laplace transform solve the initial-value problem $y' - y = 6e^{-2t}$, $y(0) = 0$.

   Show all work.

   Solution: After applying Laplace transform to both sides we get $L\left[y' - y\right] = L\left[6e^{-2t}\right]$,

   $sY(s) - Y(s) - y(0) = \dfrac{6}{s+2}$, $(s-1)Y(s) = \dfrac{6}{s+2}$, $Y(s) = \dfrac{6}{(s-1)(s+2)} = \dfrac{2}{s-1} - \dfrac{2}{s+2}$

   $y(t) = L^{-1}\left[\dfrac{2}{s-1} - \dfrac{2}{s+2}\right] = 2L^{-1}\left[\dfrac{1}{s-1}\right] - 2L^{-1}\left[\dfrac{1}{s+2}\right]$, $\quad y(t) = 2e^{t} - 2e^{-2t}$.

2. Use the Heaviside function to redefine the function

$$g(t) = \begin{cases} 1, & 0 \le t < 1 \\ \sin(2\pi t), & t \ge 1 \end{cases}$$

   then find Laplace transform of $g(t)$.

   Solution: $g(t) = 1 \cdot H_{01}(t) + \sin(2\pi t) \cdot H(t-1)$

   Due to periodicity of $\sin$ we have $\sin(2\pi t) = \sin(2\pi t - 2\pi) = \sin(2\pi(t-1))$

   Hence $g(t) = H(t) - H(t-1) + \sin(2\pi(t-1))\, H(t-1)$

   $L[g(t)] = L\left[H(t) - H(t-1) + \sin(2\pi(t-1))\, H(t-1)\right]$

   By linearity of the Laplace transform, property of $H(t)$, and the second translation formula we have

   $L[g(t)] = L\left[H(t)\right] - L\left[H(t-1)\right] + L\left[H(t-1)\sin(2\pi(t-1))\right] = L\left[H(t)\right] - e^{-s}\, L\left[1\right] + e^{-s}\, L\left[\sin(2\pi t)\right]$

   $L[g(t)] = \dfrac{1}{s} - \dfrac{e^{-s}}{s} + \dfrac{2\pi\, e^{-s}}{s^2 + 4\pi^2}$

3. Using the unit impulse response function and convolution find the solution to the initial-value problem

$$y'' - 6y' + 13y = g(t), \qquad y(0) = 0, \quad y'(0) = 4,$$

where $g(t)$ is a piecewise continuous function.

Solution: First, we find the unit impulse response function for the equation.

The characteristic equation is $s^2 - 6s + 13 = (s-3)^2 + 2^2$,

$e(t) = L^{-1}\left[\dfrac{1}{(s-3)^2+2^2}\right] = \frac{1}{2} \cdot L^{-1}\left[\dfrac{2}{(s-3)^2+2^2}\right] = \frac{1}{2}e^{3t}\sin 2t$

by the second translation formula.

Then the state-free solution is $y_s(t) = (e * g)(t) = \dfrac{1}{2}\displaystyle\int_0^t e^{3(t-u)} \sin(2(t-u))\, g(u)\, du$

$$\left[= \frac{1}{2}\int_0^t e^{3u} \sin(2u)\, g(t-u)\, du\right]$$

The imput-free solution is $y_i(t) = 4e(t) = 2e^{3t}\sin 2t$

Therefore the solution is $y(t) = 2e^{3t}\sin 2t + \dfrac{e^{3t}}{2}\displaystyle\int_0^t e^{-3u} \sin(2t-2u)\, g(u)\, du$

$\left[\text{or} \quad y(t) = 2e^{3t}\sin 2t + \dfrac{1}{2}\displaystyle\int_0^t e^{3u} \sin(2u)\, g(t-u)\, du\right]$

4. For the initial-value problem $\qquad y' = \dfrac{y}{t+1}, \quad y(0) = 1$

calculate the second iteration y_2 of Euler's method with step size $h = 0.1$. Simplify your answer.

Solution: EM: $y_{n+1} = y_n + f(t_n, y_n)h = y_n + 0.1\dfrac{y_n}{t_n+1}$

$y_0 = 1,\ t_0 = 0, \quad y_1 = y_0 + 0.1 \cdot \dfrac{y_0}{t_0+1} = 1 + 0.1 \cdot \dfrac{1}{0+1} = 1 \cdot \dfrac{11}{10} = 1.1$

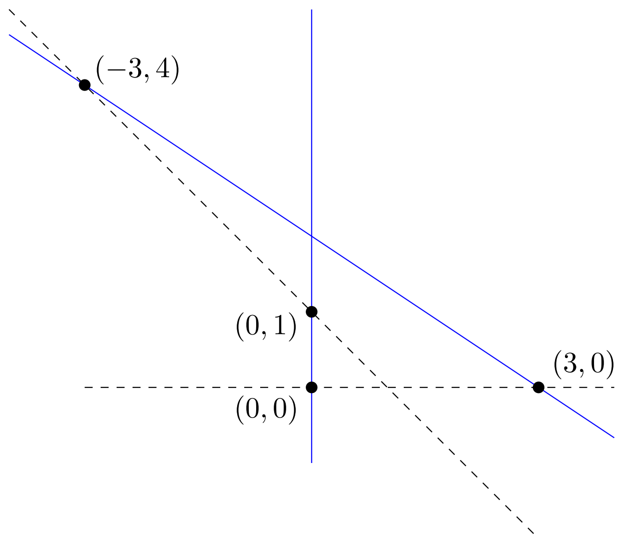
$t_1 = t_0 + h = 0.1, \quad y_2 = y_1 \cdot \dfrac{10t_1+11}{10t_1+10} = \dfrac{11}{10} \cdot \dfrac{1+11}{1+10} = \dfrac{11}{10} \cdot \dfrac{12}{11} = \dfrac{12}{10} = 1.2$

5. For the system of differential equations

$$x' = x(6 - 2x - 3y)$$
$$y' = y(1 - x - y)$$

(a) find x-nullcline and y-nullcline. Draw a plot.

Solution: x-nullcline: $x(6-2x-3y)=0 \;\Leftrightarrow\; x=0$ or $6-2x-3y=0$. Therefore, x-nullcline is a union of two lines $x=0$ and $y=-\frac{2}{3}x+2$ (solid lines on the plot below).
y-nullcline: $y(1-x-y)=0 \;\Leftrightarrow\; y=0$ or $1-x-y=0$.
Therefore, y-nullcline is a union of two lines $y=0$ and $y=-x+1$ (dashed lines).



(b) find equilibrium points. Mark them on the plot.

Solution: There are four equilibrium points $(0,0)$, $(0,1)$, $(3,0)$, and $(-3,4)$.

6. Find the general solutions to the system $y_1(t)$ and $y_2(t)$ of the system

$$\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= \;\; y_1 + y_2 \end{aligned}$$

Solution: The vector form of the system is $\bar{\mathsf{y}}' = A\bar{\mathsf{y}}$,

where $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, $\bar{\mathsf{y}}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

$T=4$, $D=3+1=4$.

Characteristic equation $\lambda^2-4\lambda+4=(\lambda-2)^2=0$ has a repeated root $\lambda=2$.

Let $\bar{\mathsf{v}}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be an eigenvector.

Then $(A-\lambda I)\,\bar{\mathsf{v}}_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We get $u_1 - u_2 = 0$. Take $u_1 = 1$ and $u_2 = 1$. Then $\bar{\mathsf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

To find $\bar{\mathsf{v}}_2$ we use the equality $(A - \lambda I)\,\bar{\mathsf{v}}_2 = \bar{\mathsf{v}}_1$

Let $\bar{\mathsf{v}}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then $(A - \lambda I)\,\bar{\mathsf{v}}_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We get $u_1 - u_2 = 1$. Take $u_1 = 2$ and $u_2 = 1$. Then $\bar{\mathsf{v}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The general solution is

$$\bar{\mathsf{y}}(t) = e^{2t}\,(C_1\,\bar{\mathsf{v}}_1 + C_2\,(\bar{\mathsf{v}}_2 + t\,\bar{\mathsf{v}}_1)) = e^{2t}\left(\begin{pmatrix} C_1 \\ C_1 \end{pmatrix} + C_2 \begin{pmatrix} 2+t \\ 1+t \end{pmatrix}\right)$$

$$\bar{\mathsf{y}}(t) = \begin{pmatrix} (C_1 + 2C_2 + C_2 t)\,e^{2t} \\ (C_1 + C_2 + C_2 t)\,e^{2t} \end{pmatrix}$$

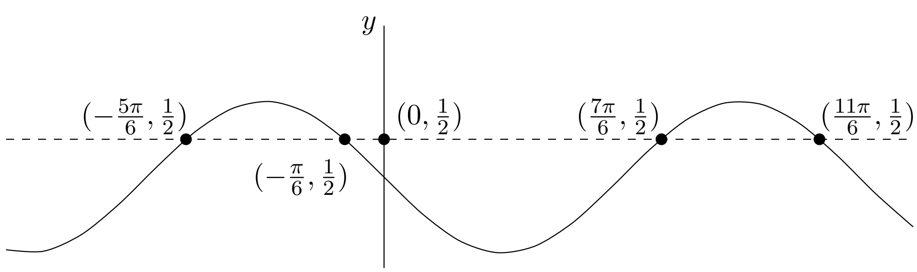
Therefore $y_1(t) = (C_1 + 2C_2 + C_2 t)\,e^{2t}$, $y_2(t) = (C_1 + C_2 + C_2 t)\,e^{2t}$

bonus problem Find all equilibrium points for the system of differential equations

$$x' = 1 - 2y$$
$$y' = x \sin x + xy$$

Solution: x-nullcline: $y = \frac{1}{2}$ (dashed line on the plot below).

y-nullcline: $x = 0$, $y = -\sin x$ (solid lines).



Equilibrium points are solutions to two systems of equations

$x = 0$, $y = \frac{1}{2}$ and $y = -\sin x$, $y = \frac{1}{2}$

The first system gives an obvious equilibrium point $\left(0, \frac{1}{2}\right)$.

The second system reduces to the single equation $\sin x = -\frac{1}{2}$.

Its solutions are $x = -\frac{\pi}{6} + 2\pi k$ and $x = \frac{7\pi}{6} + 2\pi n$, where k and n are integers.

Therefore, the equilibrium points are $\left(0, \frac{1}{2}\right)$, $\left(-\frac{\pi}{6} + 2\pi k, \frac{1}{2}\right)$, and $\left(\frac{7\pi}{6} + 2\pi n, \frac{1}{2}\right)$.