Math 0290 **Midterm Exam 1**

Spring 2017 **Solutions**

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1. A ball is dropped from the top of a cliff with the initial velocity 0 meters/second. The ball has mass of 0.1 kg. The air resistance force proportional to velocity, given that a resistance of 2 Newtons is experienced at a **downward** velocity of 4 m/sec.

   (a) Write down a differential equation that describes the ball motion and solve it.

   *Solution:* We direct the $x$-axis upward.

   Then the equation of the motion of the ball is $0.1v' = -0.1g - rv$.

   It is given that $2 = r \cdot 4 \quad \Rightarrow \quad r = 0.5$.

   Then the equation is $v' = -g - 5v$. It is a separable equation.

   $$\int \frac{dv}{v + \frac{g}{5}} = -5 \int dt, \quad \ln\left|v + \frac{g}{5}\right| = -5t + C, \quad v + \frac{g}{5} = Ae^{-5t}, \quad v(t) = Ae^{-5t} - \frac{g}{5}.$$

   The initial condition $v(0) = 0$ gives $A = \frac{g}{5} \quad \Rightarrow \quad v(t) = \frac{g}{5}\left(e^{-5t} - 1\right)$.

   (b) How long will it take the ball to reach downward velocity 1.5 m/sec? Assume that the gravity constant is 10 m/sec$^2$. Leave your answer in exact form.

   *Solution:* We have to find time $t$ when $v(t) = -1.5$. Then $\frac{g}{5} = 2$ and

   $2e^{-5t} - 2 = -1.5 \quad \Leftrightarrow \quad e^{-5t} = \frac{1}{4} \quad \Leftrightarrow \quad -5t = -\ln 4 \quad \Leftrightarrow \quad t = 0.2 \ln 4 = 0.4 \ln 2$ sec.

2. Consider the initial value problem:

   $$\frac{dy}{dx} = \frac{6x + 4}{3y^2}, \qquad y(0) = 1.$$

   (a) Find the explicit particular solution of this problem.

   *Solution:* It is a **separable** equation.

   $\frac{dy}{dx} = \frac{6x + 4}{3y^2}, \quad 3y^2\, dy = (6x + 4)\, dx, \quad \int 3y^2 dy = \int (6x + 4)\, dx, \quad y^3 = 3x^2 + 4x + C.$

   Then the general solution is $y(x) = \left(3x^2 + 4x + C\right)^{1/3}$

The initial condition gives $y(0) = C^{1/3} = 1,\ C = 1$

Therefore, the particular solution is $y(x) = \left(3x^2 + 4x + 1\right)^{1/3}$.

(b) Determine the interval of existence of your particular solution.

(Note: If you do not manage to solve (a), then assume that the solution is $y(x) = \frac{1}{9}(x^2 + 7x + 10)$).

*Solution:* From the equation we have $y(x) \neq 0$.

Therefore, $\left(3x^2 + 4x + 1\right)^{1/3} \neq 0 \quad \Leftrightarrow \quad 3x^2 + 4x + 1 \neq 0$.

$3x^2 + 4x + 1 = 0$ when $x = -1$ or $x = -\frac{1}{3}$. These points divide the $x$-axis into three subintervals $(-\infty, -1)$, $(-1, -\frac{1}{3})$, and $(-\frac{1}{3}, \infty)$.

The initial condition is defined at $x = 0$.

Therefore the interval of existence is the one that contains 0, i.e. $(-\frac{1}{3}, \infty)$.

(c) Use the Euler's method with step size $h = 1$ to derive an approximation to the particular solution at $x = 1$.

*Solution:* Euler's Method: $y_{n+1} = y_n + \dfrac{6x_n + 4}{3y_n^2}$

$y_0 = 1,\ x_0 = 0, \quad y(1) \approx y_1 = y_0 + \dfrac{6x_0 + 4}{3y_0^2} = 1 + \frac{4}{3} = \frac{7}{3} = 2\frac{1}{3}$

(d) Determine the error of the Euler's method at $x = 1$.

*Solution:* error $= |y(1) - y_1|$, $\quad y(1) = (3 + 4 + 1)^{1/3} = (8)^{1/3} = 2$.

error $= \left|2 - 2\frac{1}{3}\right| = \frac{1}{3}$

3. A tank is filled with 400 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 4 gal/min. The well-mixed solution is pumped out at a rate of 8 gal/min. Find the number of pounds of salt in the tank as a function of time.

*Solution:* Let $x(t)$ be the amount of salt in pounds in the tank at time $t$ measured in minutes.

Rate in $= 2$ lb/gal $\times$ 4 gal/min $=$ 8 lb/min.

The volume in the tank is $V(t) = 400 + (4-8)t = 400 - 4t$ gallons. Then

Rate out $= \dfrac{x}{400-4t}$ lb/gal $\times 8$ gal/min $= \dfrac{8x}{4(100-t)}$ lb/min $= \dfrac{2x}{100-t}$ lb/min

Balance equation: $\dfrac{dx}{dt} = 8 - \dfrac{2x}{100-t} \quad \Leftrightarrow \quad x' + \dfrac{2}{100-t}\,x = 8 \quad \Leftrightarrow \quad x' - \dfrac{2}{t-100}\,x = 8.$

It is a **first order linear** equation.

The integrating factor is $u = e^{-2\int \frac{1}{t-100}\,dt} = e^{-2\ln|t-100|} = (t-100)^{-2}$.

Then $((t-100)^{-2}x)' = 8(t-100)^{-2}, \quad (t-100)^{-2}x = -8(t-100)^{-1} + C,$

$x(t) = -8(t-100) + C(t-100)^2$

$x(0) = 8 \cdot 100 + C \cdot 100 \cdot 100 = 0 \quad \Leftrightarrow \quad 8 + C \cdot 100 = 0 \quad \Leftrightarrow \quad C = -0.08.$

Therefore the number of pounds of salt in the tank in $t$ minutes is

$x(t) = -8(t-100) - 0.08(t-100)^2.$

(The result can be simplified to $x(t) = 0.08t(100-t)$ ).

(Note that the result makes sense when $t < 100$ ).

4. Consider a simple RLC circuit with $R = 5\ \Omega$, $C = 0.5$ F, $E = 10$ V and there is no inductor. If the charge on the capacitor is zero at time $t = 0$, find the ensuing charge on the capacitor and the current in the circuit at time $t$. Find $I(t)$ if $I(0) = 2$ A.

*Solution:* We have the IVP $5\,\dfrac{dQ}{dt} + 2Q = 10, \quad Q(0) = 0.$

First, we simplify the equation to $\dfrac{dQ}{dt} + 0.4Q = 2.$

It is a first order linear equation. The integrating factor is $u(t) = e^{\int 0.4\,dt} = e^{0.4t}$

Then $(e^{0.4t}\,Q)' = 2\,e^{0.4t} \quad \Rightarrow \quad e^{0.4t}\,Q = 5e^{0.4t} + C \quad \Rightarrow \quad Q(t) = 5 + Ce^{-0.4t}$

[ An alternative solution: $5\dfrac{dQ}{dt} = -2(Q-5)$, $\int \dfrac{dQ}{Q-5} = -0.4\int dt$, $\ln|Q-5| = -0.4t + c$, $Q - 5 = Ce^{-0.4t}$, $Q = 5 + Ce^{-0.4t}$ ]

Then $Q(0) = 0$ gives $C = -5$ and $Q(t) = 5 - 5e^{-0.4t}$

$I(t) = \dfrac{dQ}{dt} = 2e^{-0.4t}$

5. For the equation $y'' + y' - 2y = 9e^t - 6t$.

(a) Find the fundamental set of solutions of the corresponding homogeneous equation.

*Solution:* Homogeneous equation is $y'' + y' - 2y = 0$.

Char. eq. is $\lambda^2 + \lambda - 2 = 0$. It has two distinct real roots $\lambda_1 = 1$ and $\lambda_2 = -2$ .

FSS: $y_1(t) = e^t$, $y_2(t) = e^{-2t}$.

(b) Find a particular solution by using the method of undetermined coefficients.

*Solution:* We split the forced term into two terms $9e^t$ and $-6t$ and find two trial particular solutions for each of them.

For the forced term $9e^t$ we are looking for a trial solution in the form $y_p(t) = ate^t$ because the function $e^t$ is in the FSS.

Then $y_p' = ae^t + y$ and $y_p'' = 2ae^t + y$.

$y_p'' + y_p' - 2y_p = 2ae^t + y + ae^t + y - 2y = 3ae^t = 9e^t. \quad \Rightarrow \quad a = 3.$

Hence $y_{p1}(t) = 3te^t$.

For the forced term $-6t$ we are looking for a trial solution in the form $y_p(t) = at + b$.

Then $y_p' = a$ and $y_p'' = 0$.

$y_p'' + y_p' - 2y_p = 0 + a - 2at - 2b = -6t. \quad \Rightarrow \quad a = 3,\ b = 1.5.$

Hence $y_{p2}(t) = 3t + 1.5$.

Therefore $y_p(t) = y_{p1}(t) + y_{p2}(t) = 3te^t + 3t + 1.5$.

(c) Find a particular solution by using the method of variation of parameters.

[ Hint: $\displaystyle\int te^{at}\,dt = \frac{at-1}{a^2}\,e^{at} \quad (+C)$].

*Solution:* $W(t) = e^t \cdot (-2)e^{-2t} - e^t \cdot e^{-2t} = -3e^{-t}$ and $\dfrac{1}{W} = -\dfrac{e^t}{3} = -\dfrac{1}{3} \cdot e^t$.

$$v_1 = -\int \frac{y_2\,(9e^t - 6t)}{W}\,dt = -\int \frac{1}{W} \cdot e^{-2t}(9e^t - 6t)\,dt = \int \frac{1}{3} \cdot e^t \cdot e^{-2t}(9e^t - 6t)\,dt$$

$$= \int (3 - 2te^{-t})\,dt = 3t - 2 \cdot \frac{-t-1}{(-1)^2} e^{-t} = 3t + 2te^{-t} + 2e^{-t}.$$

$$v_2 = \int \frac{y_1\,(9e^t - 6t)}{W}\,dt = \int \frac{1}{W} \cdot e^t(9e^t - 6t)\,dt = \int \left(-\frac{1}{3}\right) \cdot e^t \cdot e^t(9e^t - 6t)\,dt$$

$= \int \left(-3e^{3t} + 2te^{2t}\right)\, dt = -e^{3t} + 2 \cdot \frac{2t-1}{2^2} e^{2t} = -e^{-3t} + te^{2t} - \frac{1}{2} e^{2t}.$

Then $y_p = (3t + 2te^{-t} + 2e^{-t})\, e^t + \left(-e^{3t} + te^{2t} - \frac{1}{2} e^{2t}\right) e^{-2t} = 3te^t + 2t + 2 - e^t + t - 0.5$

$y_p = 3te^t + 3t + 1.5 - e^t.$

[ If alternatively FSS is $y_1(t) = e^{-2t}$, $y_2(t) = e^t$ then

$W(t) = e^{-2t} \cdot e^t - (-2)e^{-2t} \cdot e^t = 3e^{-t}$ and $\frac{1}{W} = \frac{e^t}{3} = \frac{1}{3} \cdot e^t.$

$v_1 = -\int \frac{y_2\,(9e^t - 6t)}{W}\, dt = -\int \frac{1}{W} \cdot e^t(9e^t - 6t)\, dt = -\int \frac{1}{3} \cdot e^t \cdot e^t(9e^t - 6t)\, dt$

$= \int \left(-3e^{3t} + 2te^{2t}\right)\, dt = -e^{3t} + 2 \cdot \frac{2t-1}{2^2} e^{2t} = -e^{-3t} + te^{2t} - \frac{1}{2} e^{2t}.$

$v_2 = \int \frac{y_1\,(9e^t - 6t)}{W}\, dt = \int \frac{1}{W} \cdot e^{-2t}(9e^t - 6t)\, dt = \int \frac{1}{3} \cdot e^t \cdot e^{-2t}(9e^t - 6t)\, dt$

$= \int (3 - 2te^{-t})\, dt = 3t - 2 \cdot \frac{-t-1}{(-1)^2} e^{-t} = 3t + 2te^{-t} + 2e^{-t}.$

Then $y_p = \left(-e^{3t} + te^{2t} - \frac{1}{2} e^{2t}\right) e^{-2t} + (3t + 2te^{-t} + 2e^{-t})\, e^t = -e^t + t - 0.5 + 3te^t + 2t + 2$

$y_p = 3te^t + 3t + 1.5 - e^t.$ ]

6. A forced mass spring system with an external driving force is modeled by

$$x'' + 2x' + 10x = 15 \sin 4t,$$

where $t$ is measured in seconds and $x$ in meters.

(a) Find the transient state, i.e. the solution to the associated homogeneous equation.

*Solution:* This is the underdamped case because $c = 1 < w_0 = \sqrt{10}$.

Homogeneous equation is $x'' + 2x' + 10x = 0$.

Char. eq. is $\lambda^2 + 2\lambda + 10 = 0$. It has complex roots $\lambda_1 = -1 + 3i$ and $\lambda_2 = -1 - 3i$ .

FSS: $x_1(t) = e^{-t} \cos 3t$, $x_2(t) = e^{-t} \sin 3t$.

The transient state is $x(t) = e^{-t}\,(c_1 \cos 3t + c_2 \sin 3t)$.

(b) Find the steady-state, i.e. a particular solution, to the forced equation. What are the amplitude and the phase of the steady-state?

*Solution:* In polar coordinates $x_p = \dfrac{A}{R(w)}\cos(wt-\phi)$ where $w=4$ and

$R = R(w) = \sqrt{(w_0^2 - w^2)^2 + 4c^2w^2} = \sqrt{(10-4^2)^2 + 4\cdot 1^2\cdot 4^2} = 10$

The amplitude of the steady-state is $\dfrac{A}{R} = \dfrac{15}{10} = 1.5$

To find $\phi$ we use $R\sin\phi = 2cw$ or $10\sin\phi = 8$ which gives $\phi = \arcsin(0.8)$.

Therefore $x_p = 1.5\cos(4t - \arcsin(0.8))$.

[An alternative solution:
$x_p(t) = a\cos 4t + b\sin 4t,\ \ x_p'(t) = -4a\sin 4t + 4b\cos 4t,\ \ x_p''(t) = -16x_p$. Then

$x_p'' + 2x_p' + 10x_p = -6x_p + 2x_p' = -6a\cos 4t - 6b\sin 4t - 8a\sin 4t + 8b\cos 4t = 15\sin 4t$

$-6a + 8b = 0,\ \ -8a - 6b = 15 \ \ \Rightarrow\ \ a = -1.2,\ b = -0.9$

$x_p(t) = -1.2\cos 4t - 0.9\sin 4t$

The amplitude is
$A = \sqrt{a^2+b^2} = \sqrt{(-1.2)^2 + (-0.9)^2} = \sqrt{1.44 + 0.81} = \sqrt{2.25} = 1.5$

$\tan\phi = \dfrac{b}{a} = \frac{-0.9}{-1.2} = \frac{9}{12} = \frac{3}{4}$. So the phase is $\phi = \tan^{-1}\left(\frac{3}{4}\right)$

Note that $\tan^{-1}\left(\frac{3}{4}\right) = \arcsin(0.8)$. You can see it from the right triangle with sides 3, 4, and 5.]

(c) Find the general solution.

*Solution:* $x(t) = e^{-t}\left(c_1\cos 3t + c_2\sin 3t\right) + 1.5\cos(4t - \arcsin(0.8))$.

bonus problem Explain the difference in particular solutions obtained in parts (b) and (c) of the problem 5 and how it may affect the solution of the differential equation $y'' + y' - 2y = 9e^t - 6t$.

*Solution:* The difference $e^t$ between particular solutions is one of the solutions in the FSS. Therefore this difference does not affect the general solution.