Math 0290 **Midterm Exam 2**

Spring 2017 **S o l u t i o n s**

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1. Results of this problem will be used in other problems. Therefore do all calculations carefully and double check them.

Find the inverse Laplace transform of the functions

(a) $Y(s) = \dfrac{2}{s^2 - 2s + 5}$

Solution: $$\frac{2}{s^2 - 2s + 5} = \frac{2}{s^2 - 2s + 1 + 4} = \frac{2}{(s-1)^2 + 2^2}$$

Therefore $L^{-1}\left[\dfrac{2}{s^2 - 2s + 5}\right] = L^{-1}\left[\dfrac{2}{(s-1)^2 + 2^2}\right] = e^t \sin 2t.$

(b) $F(s) = \dfrac{s}{(s^2 + 9)^2}$

Solution: Using the derivative of LT formula we have

$$L[t \sin 3t] = -\left(L[\sin 3t]\right)' = -\left(\frac{3}{s^2 + 9}\right)' = -\left(3(s^2+9)^{-1}\right)' = -\left(3 \cdot (-1) \cdot 2s(s^2+9)^{-2}\right)$$

$$= \frac{6s}{(s^2+9)^2} = 6F(s)$$

Therefore $L\left[\frac{1}{6} t \sin 3t\right] = F(s)$ and $L^{-1}[F(s)] = \frac{1}{6} t \sin 3t.$

2. By using Laplace transform solve the initial-value problem

$$y'' + 9y = \cos 3t, \quad y(0) = 0, \quad y'(0) = 0.$$

Do not use the convolution. [Hint: use a result from the previous problem].

Solution: After applying Laplace transform to the equation we get $(s^2+9)Y(s) = \dfrac{s}{s^2+9}$

Then $Y(s) = \dfrac{s}{(s^2+9)^2}$ and $y(t) = -\frac{1}{6} t \sin 3t$ by the previous problem.

3. Consider the initial-value problem $y' + 3y = g(t)$, $y(0) = 5$, where

$$g(t) = \begin{cases} 0, & \text{for } 0 \le t < 2 \\ 3t - 5, & \text{for } t \ge 2 \end{cases}$$

(a) Describe the function $g(t)$ in terms of the Heaviside function.

Solution: $g(t) = (3t-5)H(t-2) = (3t-6)H(t-2) + H(t-2)$

$g(t) = 3(t-2)H(t-2) + H(t-2)$

(b) By using Laplace transform solve the initial-value problem. Do not use the convolution.

Solution: $L[y' + 3y] = sY(s) - 5 + 3Y(s) = (s+3)Y(s) - 5$

$$L[g(t)] = L\left[3(t-2)H(t-2) + H(t-2)\right] = 3e^{-2s} \cdot \frac{1}{s^2} + e^{-2s} \cdot \frac{1}{s} = e^{-2s} \cdot \frac{s+3}{s^2}$$

Then $(s+3)Y(s) - 5 = e^{-2s} \cdot \dfrac{s+3}{s^2}$, $(s+3)Y(s) = e^{-2s} \cdot \dfrac{s+3}{s^2} + 5$

$$Y(s) = e^{-2s} \cdot \frac{1}{s^2} + \frac{5}{s+3}$$

$$y(t) = L^{-1}\left[e^{-2s} \cdot \frac{1}{s^2} + \frac{5}{s+3}\right] = L^{-1}\left[e^{-2s} \cdot \frac{1}{s^2}\right] + 5L^{-1}\left[\frac{1}{s+3}\right]$$

$y(t) = H(t-2)(t-2) + 5e^{-3t}$

(c) Create a piecewise definition for your solution $y(t)$ that does not contain the Heaviside function.

Solution:

$$y(t) = \begin{cases} 5e^{-3t}, & 0 \le t < 2 \\ t - 2 + 5e^{-3t}, & t \ge 2 \end{cases}$$

4. Use Laplace transform and the convolution to find a solution to the initial-value problem

$$y'' - 2y' + 5y = t^{2/3}, \qquad y(0) = 0, \quad y'(0) = -2$$

[Hint: use a result from the problem 1].

Solution: First, we find the unit impulse response function for the equation.

$$e(t) = L^{-1}\left[\frac{1}{s^2-2s+5}\right] = \frac{1}{2}\cdot L^{-1}\left[\frac{2}{s^2-2s+5}\right] = \tfrac{1}{2}e^t\sin 2t.$$

Then the state-free solution is $y_s(t) = e(t) * t^{2/3} = \frac{1}{2}\int_0^t e^u \sin 2u \cdot (t-u)^{2/3}\,du$

$$\left[= \frac{1}{2}\int_0^t e^{(t-u)}\sin 2(t-u)\cdot u^{2/3}\,du\right]$$

The input-free solution is $y_i(t) = -2e(t) = -e^t\sin 2t$

Therefore the solution is $y(t) = -e^t\sin 2t + \frac{1}{2}\int_0^t e^u\sin 2u\cdot(t-u)^{2/3}\,du$

5. For the system of differential equations
$$x' = -x+4y$$
$$y' = -2x+5y$$

(a) Find eigenvalues and eigenvectors.

Solution: It is a linear system with $A = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$

$D = \det A = -5+8 = 3,\ T = \operatorname{tr} A = -1+5 = 4,\quad T^2-4D = 4,\quad \lambda_1 = 1,\quad \lambda_2 = 3.$

$$(A-\lambda_1 I)\bar{\mathsf{v}}_1 = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \bar{0} \ \Rightarrow\ -2v_1+4v_2 = 0 \ \Leftrightarrow\ v_1 = 2v_2$$

$\Rightarrow\ \bar{\mathsf{v}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigenvector associated with the eigenvalue $\lambda_1 = 1$.

$$(A-\lambda_2 I)\bar{\mathsf{v}}_2 = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \bar{0} \ \Rightarrow\ -2v_1+2v_2 = 0 \ \Leftrightarrow\ v_1 = v_2$$

$\Rightarrow\ \bar{\mathsf{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector associated with the eigenvalue $\lambda_2 = 3$.

(b) Find the FSS and the general solution.

Solution: FSS is $\bar{\mathsf{Y}}_1 = e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix},\quad \bar{\mathsf{Y}}_2 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The general solution is $\bar{\mathsf{y}}(t) = C_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2C_1 e^t + C_2 e^{3t} \\ C_1 e^t + C_2 e^{3t} \end{bmatrix}$

(c) Find $x(t)$ and $y(t)$.

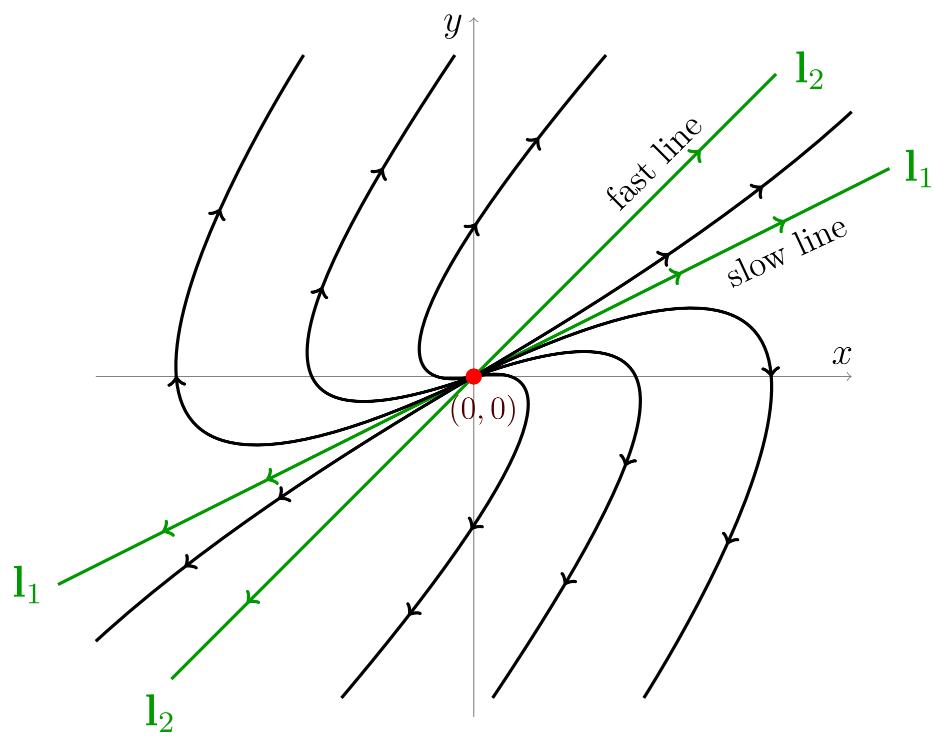
Solution: $x(t) = 2C_1 e^t + C_2 e^{3t}, \quad y(t) = C_1 e^t + C_2 e^{3t}.$

(d) Determine the type of the equilibrium point.

Solution: Eigenvalues are real of the same sign. Therefore the equilibrium point $(0, 0)$ is a nodal source.

(e) Draw the phase plane portrait.

Solution:



where l_1 is the eigenline corresponding to the eigenvector v_1 and l_2 is the eigenline corresponding to the eigenvector v_2. The line l_1 is slow and the line l_2 is fast because $\lambda_1 < \lambda_2$.

6. For the system of differential equations
$$x' = (x+2)(x-y^2)$$
$$y' = x^2 + y$$

(a) find x-nullcline and y-nullcline. Draw a plot.

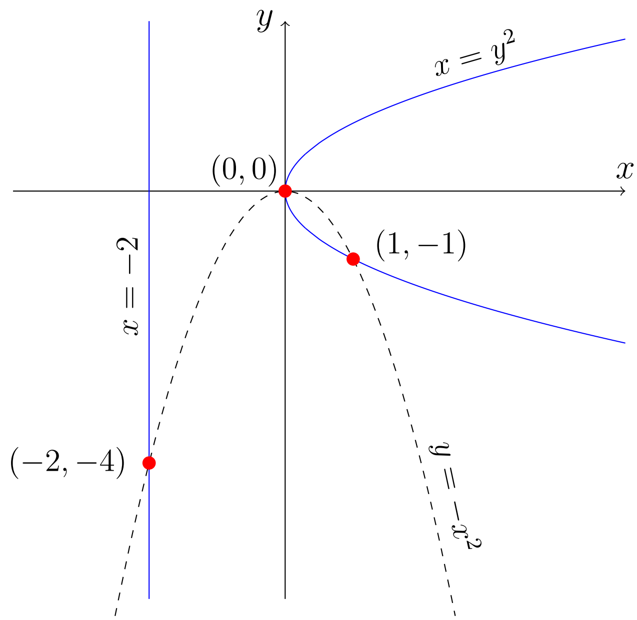
Solution: x-nullcline: $(x+2)(x-y^2)=0 \quad \Leftrightarrow \quad x=-2$ or $x=y^2$.

Therefore, x-nullcline is a union of two curves, the line $x=-2$ and the parabola $x=y^2$

(solid blue lines on the plot below).

y-nullcline: $x^2+y=0 \quad \Leftrightarrow \quad y=-x^2$.

Therefore, y-nullcline is the parabola $y=-x^2$ (dashed line).



(b) find equilibrium points. Clearly mark them on the plot.

Solution: There are three equilibrium points $(-2,-4)$, $(0,0)$, and $(1,-1)$.

(c) Determine the type of the most right equilibrium point (saddle point, nodal sink, nodal source, spiral sink, or spiral source). If the type cannot be determined, explain why. Do not draw the phase portrait.

Solution: The most right equilibrium point is $(1,-1)$.

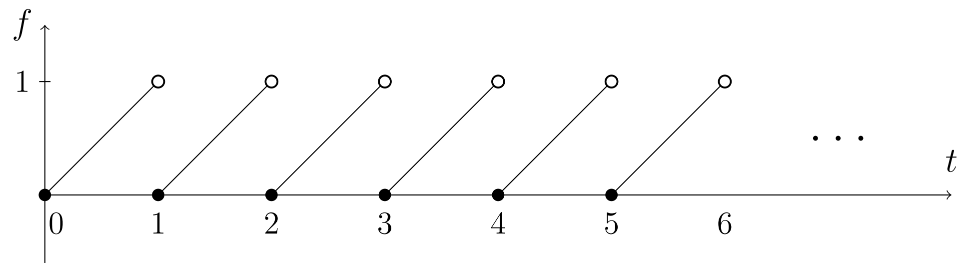
$$J=\begin{bmatrix} 2x+2-y^2 & -2xy-4y \\ 2x & 1 \end{bmatrix} \qquad J(1,-1)=\begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$$

$D = \det J = 3 - 12 = -9 < 0 \quad \Rightarrow \quad$ the linearization has a saddle.

It is generic and the type of the equilibrium point preserves for the given non-linear system.

Therefore $(1, -1)$ is a saddle point.

bonus problem Find the Laplace transform of the periodic function $f(t)$ defined for $t \in [0, \infty)$ graph of which on the interval $t \in [0, 6)$ is given below



Solution: $$f(t) = \begin{cases} t, & \text{for } 0 \le t < 1 \\ t-1, & \text{for } 1 \le t < 2 \\ t-2, & \text{for } 2 \le t < 3 \\ t-3, & \text{for } 3 \le t < 4 \\ \quad \vdots & \end{cases}$$

$$f(t) = tH_{0,1} + (t-1)H_{1,2} + (t-2)H_{2,3} + (t-3)H_{3,4} + \cdots$$

$$= t\,(H_0 - H_1) + (t-1)\,(H_1 - H_2) + (t-2)\,(H_2 - H_3) + (t-3)\,(H_3 - H_4) + \cdots$$

$$= t\,(H_0 - H_1) + t\,(H_1 - H_2) - 1\,(H_1 - H_2) + t\,(H_2 - H_3) - 2\,(H_2 - H_3)$$

$$+\, t\,(H_3 - H_4) - 3\,(H_3 - H_4) + \cdots$$

$$= t\,(H_0 - H_1 + H_1 - H_2 + H_2 - H_3 + H_3 \pm \cdots)$$

$$+\,(-H_1 + H_2 - 2H_2 + 2H_3 - 3H_3 + 3H_4 - 4H_4 \pm \cdots)$$

$$f(t) = t - (H_1 + H_2 + H_3 + H_4 + \cdots)$$

$$F(s) = L[f(t)](s) = \frac{1}{s^2} - \frac{1}{s}\,(e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + \cdots)$$

$s > 0 \quad \Rightarrow \quad 0 < e^{-ns} < 1$, for any positive integer n. Denote $r = e^{-s}$. Then $e^{-ns} = r^n$.

$$e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + \cdots = r + r^2 + r^3 + r^4 + \cdots = \frac{r}{1-r} = \frac{e^{-s}}{1-e^{-s}} \quad \text{(geometric series).}$$

Therefore $F(s) = \dfrac{1}{s^2} - \dfrac{1}{s} \cdot \dfrac{e^{-s}}{1-e^{-s}}$