Math 0290 **Final Exam**

Spring 2017 **Name:** ______________________

No calculators, no books. Show all your work (no work = no credit).
Write neatly. Simplify your answers when possible.

---

1. Consider the following ordinary differential equation (ODE) $y' + \frac{2}{x}\,y = \frac{\sin x}{x^2}$

   (a) (5 points) Find the explicit general solution to this ODE.

   (b) (5 points) Find the particular solution that satisfies $y(\pi) = 0$.

   (c) (5 points) Find the interval of existence on which the particular solution with $y(\pi) = 0$ is valid.

2. The ODE for the displacement of a spring, which is undamped and forced, takes the form

$$y'' + y = 2\sin t.$$

(a) (5 points) Find the transient state, i.e. the solution to the associated homogeneous equation.

(b) (5 points) Find the steady-state, i.e. a particular solution, to the forced equation.

(c) (5 points) Find the general solution.

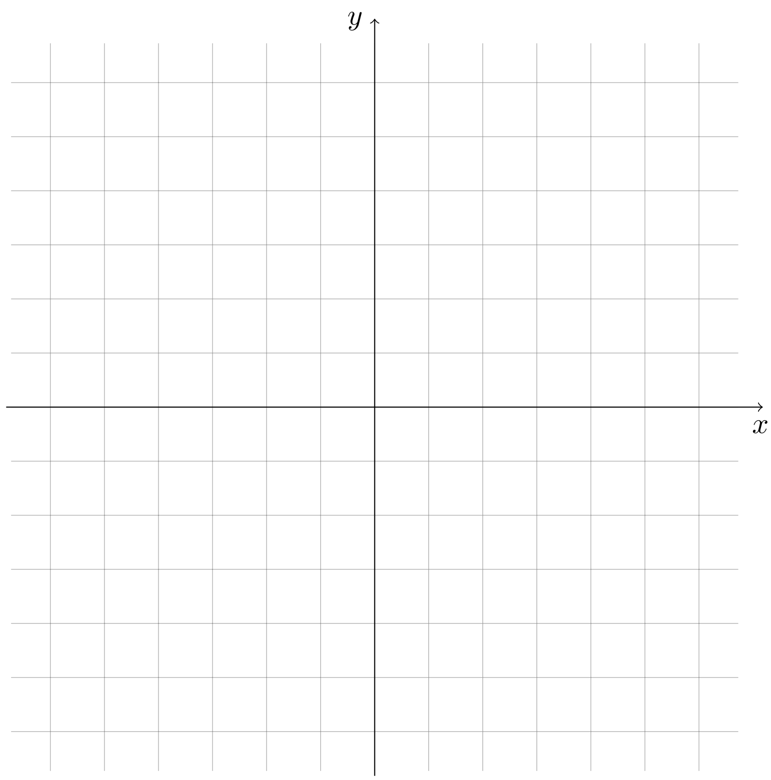
3. (15 points) Find the solution to the previous ODE $y'' + y = 2\sin t$ that satisfy the initial conditions $y(0) = 0,\ y'(0) = -1$. **Use Laplace transform**.

$$\left[\text{ Hint: } \frac{2}{(s^2+1)^2} - \frac{1}{s^2+1} = \frac{1-s^2}{(s^2+1)^2} = \frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

4. For the system of nonlinear differential equations

$$x' = x(3 - x - y)$$
$$y' = x - 2y$$

(a) (5 points) Find x-nullcline and y-nullcline. Draw a plot.

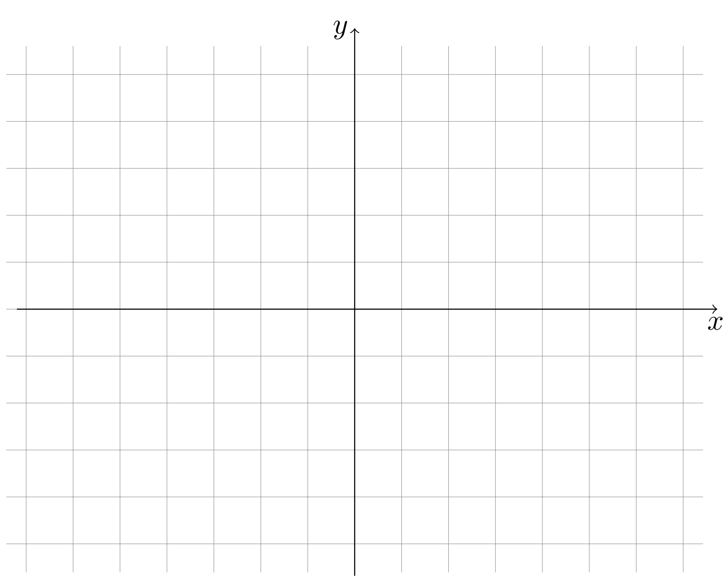


(b) (5 points) Find equilibrium points. Clearly mark them on the plot.

Equilibrium points are:

(c) (5 points) Find Jacobian matrix at every equilibrium point. Determine types of all equilibrium points of the given nonlinear system. If the type cannot be determined, explain why.

(d) (5 points) For the most left equilibrium point find eigenvalues of the corresponding linear system and its eigenvectors. Draw the local phase portrait near the equilibrium point.



5. (15 points) Expand the function $f(x) = |x|, \ -1 \le x \le 1$ in a Fourier series.

6. Given the equation $u_t(x,t) = u_{xx}(x,t) + \frac{1}{t} \cdot u(x,t), \qquad 0 < x < \pi$

with boundary conditions $u(0,t) = 0, \;\; u(\pi,t) = 0$

(a) (5 points) perform separation of variables, find the ODEs for $T(t)$ and $X(x)$.

(b) (5 points) Find boundary conditions for $X(x)$.

(c) (5 points) From the ODE for $X(x)$, find the eigenvalues λ_n and the eigenfunctions $X_n(x)$.

(d) (5 points) From the ODE for $T(t)$ find $T_n(t)$ that corresponds to λ_n.

bonus problem (10 points) Find Fourier Series expansion of the Dirac's delta function $\delta(x)$. You can assume that $L = \pi$.