Math 0290 **Final Exam**

Spring 2017 **S o l u t i o n s**

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1. Consider the following ordinary differential equation (ODE) $y' + \frac{2}{x}\,y = \frac{\sin x}{x^2}$

(a) Find the explicit general solution to this ODE.

Solution: It is a linear equation. The integrating factor is $u = e^{\int \frac{2}{x}\,dx} = x^2$

$x^2 y' + 2xy = \sin x, \quad (x^2 y)' = \sin x, \quad x^2 y = \int \sin x\,dx = -\cos x + c.$

Therefore, $y = -\frac{\cos x}{x^2} + \frac{c}{x^2} = \frac{c - \cos x}{x^2}.$

(b) Find the particular solution that satisfies $y(\pi) = 0$.

Solution: $c - \cos\pi = 0, \; c = \cos\pi = -1.$

Therefore, $y = -\frac{1 + \cos x}{x^2}.$

(c) Find the interval of existence on which the particular solution with $y(\pi) = 0$ is valid.

Solution: $x \neq 0$ and $x = \pi$ is in the interval.

Therefore the interval of existence is $(0, \infty)$.

2. The ODE for the displacement of a spring, which is undamped and forced, takes the form

$$y'' + y = 2\sin t.$$

(a) Find the transient state, i.e. the solution to the associated homogeneous equation.

Solution: Homogeneous equation is $y'' + y = 0$.

Char. eq. is $\lambda^2 + 1 = 0$. It has complex (pure imaginary) roots $\lambda = i$ and $\bar{\lambda} = -i$.

FSS: $y_1(t) = \cos t, \; y_2(t) = \sin t.$

The transient state is $y(t) = c_1 \cos t + c_2 \sin t.$

(b) Find the steady-state, i.e. a particular solution, to the forced equation.

Solution: Because the forcing term coincide with one of the fundamental solutions the trial particular solution is

$y_p(t) = t(a\cos t + b\sin t) = tv,\quad \text{where}\quad v = a\cos t + b\sin t.$

Note that $v' = -a\sin t + b\cos t$ and $v'' = -v$.

$y_p' = v + tv',\quad y_p''(t) = v' + v' + tv'' = 2v' - tv.$ Then

$y_p'' + y_p = 2v' - tv + tv = 2v' = -2a\sin t + 2b\cos t = 2\sin t \quad\Rightarrow\quad a = -1,\ b = 0.$

$y_p(t) = -t\cos t.$

(c) Find the general solution.

Solution: The general solution is $y(t) = c_1\cos t + c_2\sin t - t\cos t.$

3. Find the solution to the previous ODE $y'' + y = 2\sin t$ that satisfy the initial conditions $y(0) = 0,\ y'(0) = -1$. Use Laplace transform.

$$\left[\ \text{Hint: } \frac{2}{(s^2+1)^2} - \frac{1}{s^2+1} = \frac{1-s^2}{(s^2+1)^2} = \frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

Solution: $L[y''+y] = L[2\sin t],\quad (s^2+1)Y(s) + 1 = \dfrac{2}{s^2+1},\quad Y(s) = \dfrac{2}{(s^2+1)^2} - \dfrac{1}{s^2+1}.$

$$Y(s) = \frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

Therefore, $y(t) = -t\cos t$ (by using the derivative of Laplace transform formula).

4. For the system of nonlinear differential equations

$$x' = x(3 - x - y)$$
$$y' = x - 2y$$

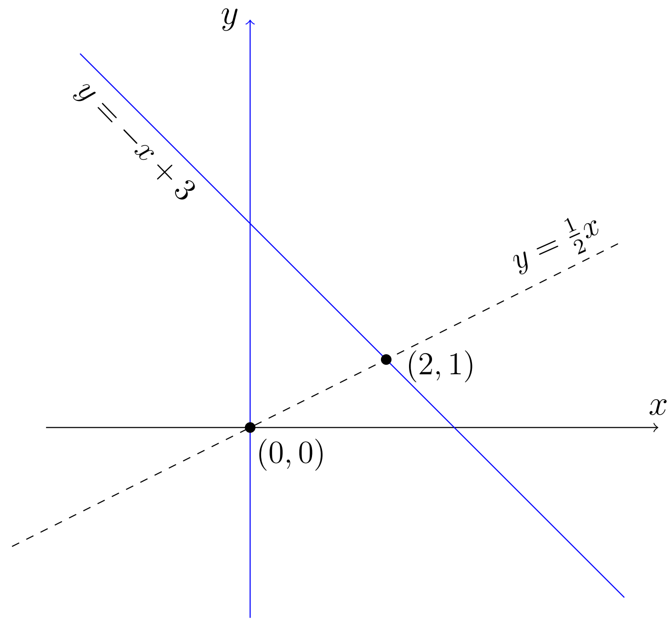
(a) Find x-nullcline and y-nullcline. Draw a plot.

Solution: x-nullcline: $x(3-x-y) = 0 \quad\Leftrightarrow\quad x = 0$ or $y = -x + 3$.

Therefore, x-nullcline is a union of two lines $x = 0$ and $y = -x + 3$ (solid blue lines on the plot below).

y-nullcline: $x - 2y = 0 \quad\Leftrightarrow\quad y = \frac{1}{2}x.$

Therefore, y-nullcline is the line $y = \frac{1}{2}x$ (dashed line).



(b) Find equilibrium points. Clearly mark them on the plot.

Solution: There are two equilibrium points $(0, 0)$ and $(2, 1)$.

(c) Find Jacobian matrix at every equilibrium point. Determine types of all equilibrium points of the given nonlinear system. If the type cannot be determined, explain why.

Solution: $J = \begin{bmatrix} 3 - 2x - y & -x \\ 1 & -2 \end{bmatrix}$

At $(0, 0)$: $J(0, 0) = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

$D = \det J = -6 < 0, \Rightarrow$ the linearization of the given system has a saddle at $(0, 0)$. This type is generic. Therefore the nonlinear system has a saddle at $(0, 0)$.

At $(2, 1)$: $J(2, 1) = \begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix}$

$D = \det J = 4 + 2 = 6 > 0, \quad T = -4 < 0, \quad T^2 - 4D = 16 - 24 = -8 < 0 \quad \Rightarrow$ the linearization of the given system has a spiral sink at $(2, 1)$. This type is generic. Therefore the nonlinear system has a spiral sink at $(2, 1)$.

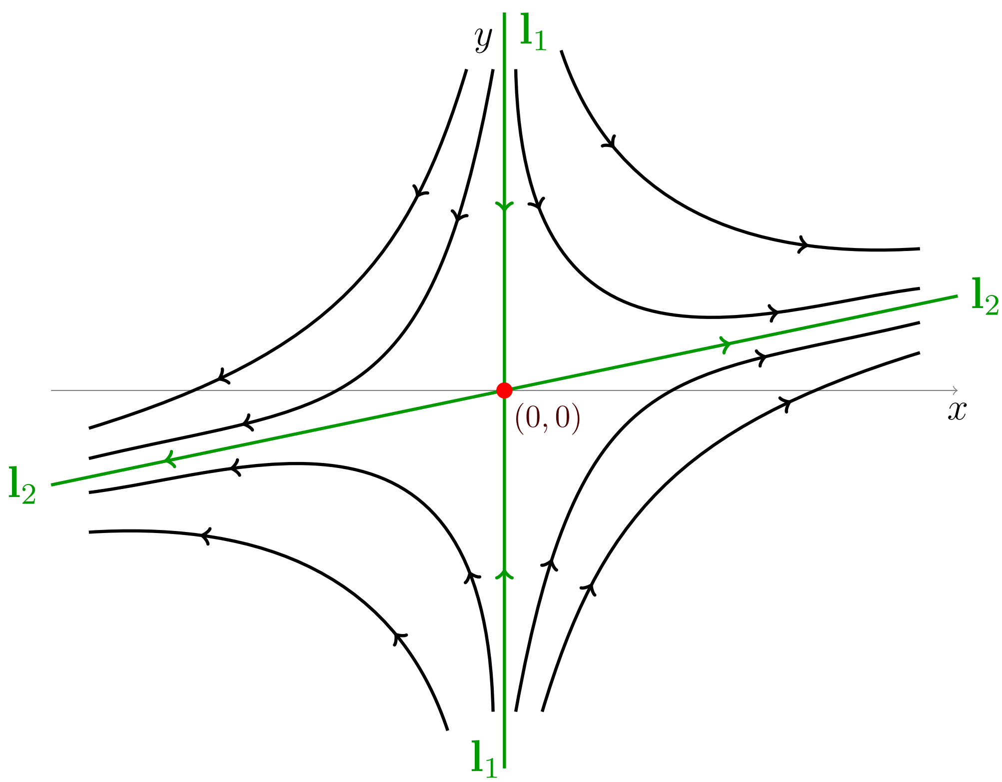
(d) For the most left equilibrium point find eigenvalues of the corresponding linear system and its eigenvectors. Draw the local phase portrait near the equilibrium point.

Solution: The most left equilibrium point is $(0,0)$.

$T = 1 > 0,\ T^2 - 4D = 1 + 24 = 25 \ \Rightarrow\ \lambda_1 = -2,\ \ \lambda_2 = 3.$

$$(J - \lambda_1 I)\bar{\mathsf{v}}_1 = \begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \bar{0} \ \Rightarrow\ v_1 = 0. \text{ Take } v_2 = 1 \ \Rightarrow\ \bar{\mathsf{v}}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$(J - \lambda_2 I)\bar{\mathsf{v}}_2 = \begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \bar{0} \ \Rightarrow\ v_1 - 5v_2 = 0 \ \Leftrightarrow\ v_1 = 5v_2 \ \Rightarrow\ \bar{\mathsf{v}}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$



5. Expand the function $f(x) = |x|,\ -1 \le x \le 1$ in a Fourier series.

Solution: $f(x)$ is even. Therefore all $b_n = 0$ and

$$a_0 = 2\int_0^1 x\,dx = 1,\ \ a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{\pi^2} \cdot \frac{(-1)^n - 1}{n^2}.$$

Then $|x| = \dfrac{1}{2} + \dfrac{2}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^n - 1}{n^2} \cos(n\pi x)$.

Alternative answer is $|x| = \dfrac{1}{2} - \dfrac{4}{\pi^2} \sum_{k=0}^{\infty} \dfrac{1}{(2k+1)^2} \cos((2k+1)\pi x)$.

6. Given the equation

$$u_t(x,t) = u_{xx}(x,t) + \frac{1}{t} \cdot u(x,t), \qquad 0 < x < \pi$$

with boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

(a) perform separation of variables, find the ODEs for $T(t)$ and $X(x)$.

Solution: $u = XT,\ u_t = XT',\ u_{xx} = X''T$

$XT' = X''T + \dfrac{1}{t} \cdot XT,\quad X\left(T' - \dfrac{1}{t} \cdot T\right) = X''T,\quad \dfrac{T'}{T} - \dfrac{1}{t} = \dfrac{X''}{X} = -\lambda = -\omega^2.$

Therefore, $\dfrac{T'}{T} = \dfrac{1}{t} - \omega^2,\quad X'' + \omega^2 X = 0.$

(b) Find boundary conditions for $X(x)$.

Solution: $X(0) = 0,\ X(\pi) = 0.$

(c) From the ODE for $X(x)$, find the eigenvalues λ_n and the eigenfunctions $X_n(x)$.

Solution: $X(x) = a\cos\omega x + b\cos\omega x.$

$X(0) = a = 0,\ X(x) = b\cos\omega x,\ X(\pi) = b\cos\omega\pi = 0,\ \omega\pi = n\pi,\ \omega_n = n.$

The eigenvalues are $\lambda_n = n^2$.

The eigenfunctions are $X_n(x) = \cos nx$ (or $b_n \cos nx$).

(d) Solve the ODE for $T(t)$ to find $T_n(t)$ that corresponds to λ_n.

Solution: $\dfrac{T'}{T} = \dfrac{1}{t} - n^2.$

It is a separable equation. $\ln T = \ln t - n^2 t + c,\quad T_n(t) = Ate^{-n^2 t}$

bonus problem Find Fourier Series expansion of the Dirac's delta function $\delta(x)$.
You can assume that $L = \pi$.

Solution: $\displaystyle\int_{-a}^{a} \delta(x)\,dx = 1,\ \int_{-a}^{a} \delta(x) f(x)\,dx = f(0)$ for any $a > 0$

by the property of the delta function.

Therefore, $\displaystyle b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin(nx)\,dx = \frac{1}{\pi}\sin(0) = 0.$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\,dx = \frac{1}{\pi}\cdot 1 = \frac{1}{\pi}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos(nx)\,dx = \frac{1}{\pi}\cos(0) = \frac{1}{\pi}.$$

Then $\displaystyle\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(nx).$