0290 DE Final Exam **Name:**

(9 problems, 100 points)

functions: t^n, e^{at}, $e^{at}\cos bt$, $t^n a^{at}$, 1, $\sin at$, $e^{at}\sin bt$, $\cos at$

Laplace transforms: $\dfrac{n!}{(s-a)^{n+1}}$, $\dfrac{s}{s^2+a^2}$, $\dfrac{1}{s-a}$, $\dfrac{s-a}{(s-a)^2+b^2}$, $\dfrac{n!}{s^{n+1}}$, $\dfrac{a}{s^2+a^2}$, $\dfrac{1}{s}$, $\dfrac{b}{(s-a)^2+b^2}$

Useful formulas: $L[e^{ct}f(t)](s) = F(s-c)$, $L[t^n f(t)](s) = (-1)^n F^{(n)}(s)$,
and $L[H(t-c)f(t-c)](s) = e^{-cs}F(s)$.

1. (10 points) The given equation is not exact. Multiply it by the given integrating factor, check that the obtained equation is exact and solve it:

$3(y+1)\,dx - 2x\,dy = 0, \quad \mu(x,y) = \dfrac{y+1}{x^4}.$

2. (15 points) For the given nonlinear system

$$x' = 8x - y^2$$
$$y' = -6y + 6x^2$$

(a) find both equilibrium points (one of them has x=2),
(b) use the Jacobian to classify each equilibrium point (saddle, spiral sink, etc.). Determine stability.

3. (10 points) For the given second order differential equation find

(a) the characteristic equation and its roots,
(b) the fundamental set of real-valued solutions (FSS). Prove that these two solutions are linearly independent on the interval of their existence (use Wronskian) and hence they form FSS,
(c) the general real-valued solution

$$y'' - 6y' + 45y = 0$$

4. (10 points) Find the general solution to the equation

$$y'' + y = \cos^2 t$$

5. (5 points) Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned}
&u_t = 5\,u_{xx}, \quad \text{for } \ t > 0, \ \ 0 < x < 2\pi, \\
&u(0,t) = u(L,t) = 0, \quad \text{for } \ t > 0, \\
&u(x,0) = -4\sin\,2x + 7\sin\,3x, \quad \text{for } \ 0 < x < 2\pi
\end{aligned}$$

(You may use the expressions $-\dfrac{n^2\pi^2 kt}{L^2}$ and $\dfrac{n\pi x}{L}$. Note: $b_2 = 0$ since L is not π).

6. (15 points) For the system

$$\begin{aligned} y_1' &= 8y_1 + 3y_2 \\ y_2' &= -6y_1 - y_2 \end{aligned}$$

find the type of the equilibrium point using TD diagram and determine is it stable, unstable or asymptotically stable. Find the eigenvalues. Associated eigenvectors are $(1, -2)^T$ and $(1, -1)^T$. Sketch the phase portrait.

7. (10 points) Use LT to solve IVP:

$$y' + y = f(t), \quad y(0) = 0, \qquad \text{where} \qquad f(t) = \begin{cases} 0, & \text{for} \quad 0 \le t < 1 \\ 5, & \text{for} \quad t \ge 1 \end{cases}$$

8. (10 points) Find the general solution of the system

$$\begin{aligned} y_1' &= -5y_1 + y_2 \\ y_2' &= -2y_1 - 2y_2 \end{aligned}$$

9. (15 points) Use the LT to solve the IVP $\quad y'' + 16y = \cos 4t, \quad y(0) = 0, \quad y'(0) = 8$

Bonus problem: (5 points) Find LT of $f(t) = \sin(3.5t)\cos(3.5t)\, e^{3.5t}$.