0290 DE Final Exam **Name:**

(10 problems, 100 points)

functions: t^n, e^{at}, $e^{at}\cos bt$, $t^n a^{at}$, 1, $\sin at$, $e^{at}\sin bt$, $\cos at$

Laplace transforms: $\frac{n!}{(s-a)^{n+1}}$, $\frac{s}{s^2+a^2}$, $\frac{1}{s-a}$, $\frac{s-a}{(s-a)^2+b^2}$, $\frac{n!}{s^{n+1}}$, $\frac{a}{s^2+a^2}$, $\frac{1}{s}$, $\frac{b}{(s-a)^2+b^2}$

Useful formulas: $L[e^{ct}f(t)](s) = F(s-c)$, $L[t^n f(t)](s) = (-1)^n F^{(n)}(s)$,
and $L[H(t-c)f(t-c)](s) = e^{-cs}F(s)$.

1. (12 points) Determine whether the given equation is exact. If it is exact, solve it:

$$(\sin\, y - y\sin\, x)dx + (\cos\, x + x\cos\, y - y)dy = 0$$

2. (9 points) For the given nonlinear system

$$x' = 8x - y^2$$
$$y' = -6y + 6x^2$$

(a) find both equilibrium points
(b) use the Jacobian to classify each equilibrium point (saddle, spiral sink, etc.)

3. (8 points) Find inverse LT of $F(s) = \dfrac{1 - e^{-s}}{s^2}$

4. (10 points) For the given second order differential equation find

(a) the characteristic equation and its roots,
(b) the fundamental set of real-valued solutions (FSS). Prove that these two solutions are linearly independent on the interval of their existence (use Wronskian) and hence they form FSS,
(c) the general real-valued solution

$$x'' - 4x' + 53x = 0$$

5. (12 points) Find the general solution to the equation

$$y'' + 6y' + 8y = 3e^{-2t} + 2t$$

6. (5 points) Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} &u_t = 4\,u_{xx}, \quad \text{for } t > 0, \ \ 0 < x < \pi, \\ &u(0,t) = u(L,t) = 0, \quad \text{for } t > 0, \\ &u(x,0) = \sin\, 2x - \sin\, 4x, \quad \text{for } 0 < x < L \end{aligned}$$

(Hints: (1) This is problem 6 from section 13.2. (2) Show that the steady-state temperature is constant, find this constant. (3) The initial temperature is represented as the Fourier series with two terms and hence the solution will contain two terms only, not infinitely many.)

7. (12 points) Use the LT to solve the IVP $\qquad y'' + y = \cos t, \quad y(0) = 0, \quad y'(0) = 0$

8. (12 points) Find a FSS for the system

$$\begin{aligned} y_1' &= 2y_1 \\ y_2' &= -6y_1 - 2y_2 \end{aligned}$$

9. (10 points) Find the convolution of the two functions $f(t) = t$ and $g(t) = e^t$.

10. (10 points) For the system

$$y_1' = 0.1y_1 + 2y_2$$
$$y_2' = -2y_1 + 0.1y_2$$

find the type of an equilibrium point and determine is it stable, unstable or asymptotically stable.

Bonus problem: (6 points) Find LT of $f(t) = \sin(3.5t)\cos(3.5t)\,e^{3.5t}$.