0290 DE Final Exam **Name:** Solutions

(9 problems, 100 points)

functions: t^n, e^{at}, $e^{at}\cos bt$, $t^n a^{at}$, 1, $\sin at$, $e^{at}\sin bt$, $\cos at$

Laplace transforms: $\frac{n!}{(s-a)^{n+1}}$, $\frac{s}{s^2+a^2}$, $\frac{1}{s-a}$, $\frac{s-a}{(s-a)^2+b^2}$, $\frac{n!}{s^{n+1}}$, $\frac{a}{s^2+a^2}$, $\frac{1}{s}$, $\frac{b}{(s-a)^2+b^2}$

Useful formulas: $L[e^{ct}f(t)](s) = F(s-c)$, $L[t^n f(t)](s) = (-1)^n F^{(n)}(s)$,
and $L[H(t-c)f(t-c)](s) = e^{-cs}F(s)$.

1. (10 points) The given equation is not exact. Multiply it by the given integrating factor, check that the obtained equation is exact and solve it:

$3(y+1)\,dx - 2x\,dy = 0, \quad \mu(x,y) = \frac{y+1}{x^4}.$

After multiplication: $3(y+1)^2 x^{-4}dx - 2(y+1)x^{-3}dy = 0$

$P = 3(y+1)^2 x^{-4}$, $\quad \frac{\partial P}{\partial y} = 6(y+1)x^{-4}$

$Q = -2(y+1)x^{-3}$, $\quad \frac{\partial Q}{\partial x} = -2(-3)(y+1)x^{-4} = 6(y+1)x^{-4}$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \text{exact.}$$

$F = \int 3(y+1)^2 x^{-4}dx = 3(y+1)^2 \int x^{-4}dx = 3(y+1)^2\left(-\frac{1}{3}\right)x^{-3} +$
$+ \varphi(y) = -(y+1)^2 x^{-3} + \varphi(y)$

$F_y = -2(y+1)x^{-3} + \varphi'(y) = Q = -2(y+1)x^{-3}$

$\Rightarrow \varphi'(y) = 0 \Rightarrow \varphi(y) = C_1$ (a constant)

Solution $F = C_2$, $\quad -\frac{(y+1)^2}{x^3} + C_1 = C_2 \Leftrightarrow \boxed{\frac{(y+1)^2}{x^3} = C}$

(C_2 is an arbitrary constant) where $C = C_1 - C_2$

2. (15 points) For the given nonlinear system

$$x' = 8x - y^2$$
$$y' = -6y + 6x^2$$

(a) find both equilibrium points (one of them has x=2),
(b) use the Jacobian to classify each equilibrium point (saddle, spiral sink, etc.). Determine stability.

The same as #2 in the other exam.

3. (10 points) For the given second order differential equation find

(a) the characteristic equation and its roots,
(b) the fundamental set of real-valued solutions (FSS). Prove that these two solutions are linearly independent on the interval of their existence (use Wronskian) and hence they form FSS,
(c) the general real-valued solution

$$y'' - 6y' + 45y = 0$$

(a) $\lambda^2 - 6\lambda + 45 = 0$, $\lambda = 3 \pm 6i$

(b) $y_1 = e^{3t}\cos 6t$, $y_2 = e^{3t}\sin 6t$

$y_1' = e^{3t}(3\cos 6t - 6\sin 6t)$, $y_2' = e^{3t}(3\sin 6t + 6\cos 6t)$

$W\big|_{t=0} = \begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix} = 6 \neq 0$ $\Rightarrow$ y_1 & y_2 are LI

$\Rightarrow$ they form FSS

interval of existence: $(-\infty, \infty)$.

(c) $\boxed{y(t) = e^{3t}(C_1 \cos 6t + C_2 \sin 6t)}$

4. (10 points) Find the general solution to the equation

$$y'' + y = \cos^2 t$$

Char eq: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$y_1 = \cos t$, $y_2 = \sin t$, $W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

$v_1 = \int -\cos^2 t \sin t \, dt = [u = \cos t, du = -\sin t dt] = \int u^2 du =$

$= \frac{u^3}{3} = \frac{1}{3}\cos^3 t$

$v_2 = \int \cos^2 t \cdot \cos t \, dt = [u = \sin t, du = \cos t dt] = \int (1-u^2) du$

$= u - \frac{1}{3}u^3 = \sin t - \frac{1}{3}\sin^3 t$

$y_p = y_1 v_1 + y_2 v_2 = \frac{1}{3}\cos^4 t + \sin^2 t - \frac{1}{3}\sin^4 t =$

$= \frac{1}{3}(\cos^2 t - \sin^2 t)(\cos^2 t + \sin^2 t) + \sin^2 t = \frac{1}{3}\cos^2 t - \frac{1}{3}\sin^2 t$

$+ \sin^2 t = \frac{1}{3}\cos^2 t + \frac{2}{3}\sin^2 t = \frac{1}{3}(\cos^2 t + 2\sin^2 t) =$

$= \frac{1}{3}(\cos^2 t + \sin^2 t + \sin^2 t) = \frac{1}{3}(1 + \sin^2 t)$

$y_p = \frac{1}{3}(1 + \sin^2 t)$

$W(t) = 1 \neq 0$ for any t. Hence y_1 and y_2 are LI and form the FSS.

The gen. sln: $\boxed{y(t) = C_1 \cos t + C_2 \sin t + \frac{1}{3} + \frac{1}{3}\sin^2 t}$

5. (5 points) Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$u_t = 5\,u_{xx}, \quad \text{for } t>0,\ 0<x<2\pi,$$
$$u(0,t) = u(L,t) = 0, \quad \text{for } t>0,$$
$$u(x,0) = -4\sin 2x + 7\sin 3x, \quad \text{for } 0<x<2\pi$$

$k=5$

$\Rightarrow L = 2\pi$

(You may use the expressions $-\frac{n^2\pi^2 kt}{L^2}$ and $\frac{n\pi x}{L}$. Note: $b_2 = 0$ since L is not π).

Steady-state sln $u_s(x)$ satisfies: $\frac{\partial^2 u_s}{\partial x^2} = 0$

$u_s(0) = u_s(2\pi) = 0$

$\frac{\partial^2 u_s}{\partial x^2} = 0 \Rightarrow u_s = ax + b.$ $\quad u_s(0) = b = 0$

$u_s(2\pi) = 2\pi a + b = 0 \Rightarrow a = 0.$

Hence $u_s(x) = 0.$

$$u(x,t) = u_s(x) + v(x,t) = v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{5}{4}n^2 t} \sin\left(\frac{nx}{2}\right)$$

since $-\frac{n^2\pi^2 kt}{L^2} = -\frac{n^2\pi^2\cdot 5t}{(2\pi)^2} = -\frac{5}{4}n^2 t$

$v(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{nx}{2}$, $\quad v(x,0) = g(x) = f(x) - u_s(x)$

$= -4\sin 2x + 7\sin 3x$

Hence $\sum_{n=1}^{\infty} b_n \sin\frac{nx}{2} = b_1 \sin\frac{x}{2} + b_2 \sin x + b_3 \sin\frac{3}{2}x$

$+ b_4 \sin 2x + b_5 \sin\frac{5}{2}x + b_6 \sin 3x + \cdots = -4\sin 2x + 7\sin 3x$

By the comparison of the coefficients we obtain

$b_n = 0$ for any n except $n=4$, $b_4 = -4$, $-\frac{5}{4}n^2 t = -20t$

$n=6$, $b_6 = 7$, $-\frac{5}{4}n^2 t = -45t$

Hence $\boxed{u(x,t) = -4e^{-20t}\sin 2x + 7e^{-45t}\sin 3x}$

5

6. (15 points) For the system

$$y_1' = 8y_1 + 3y_2$$
$$y_2' = -6y_1 - y_2$$

find the type of the equilibrium point using TD diagram and determine is it stable, unstable or asymptotically stable. Find the eigenvalues. Associated eigenvectors are $(1,-2)^T$ and $(1,-1)^T$. Sketch the phase portrait.

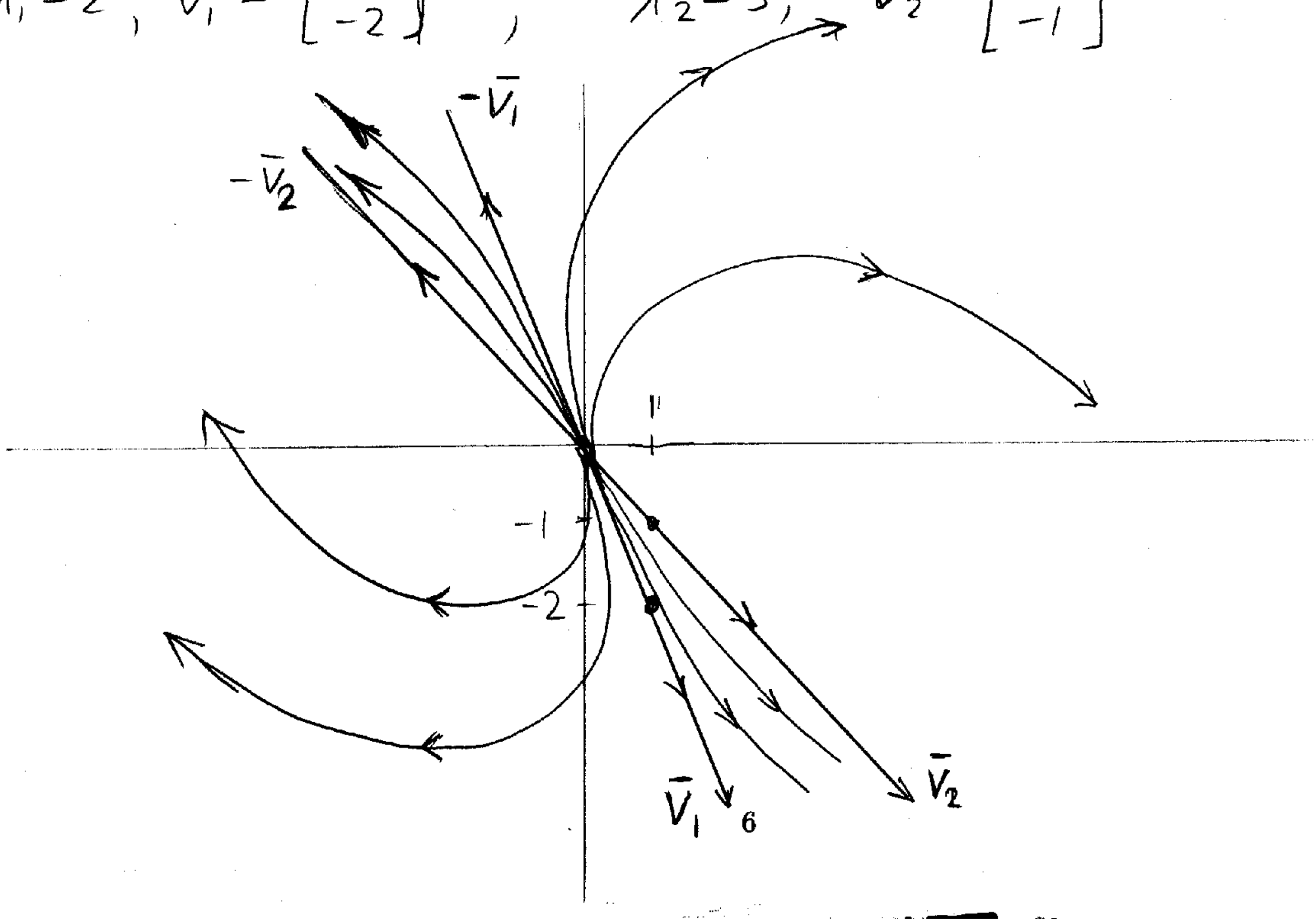
Let $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$. Then $\bar{y}' = A\bar{y}$

$T = 8 - 1 = 7$, $D = -8 + 18 = 10$, $T^2 - 4D = 49 - 40 = 9 > 0$

$\Rightarrow$ Nodal Source

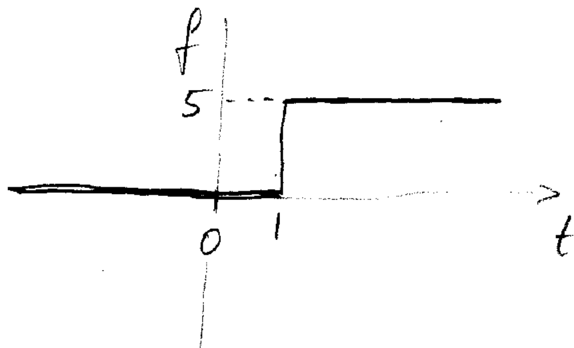
$\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4D}\right] = \frac{1}{2}[7 \pm 3]$, $\lambda_1 = 2$, $\lambda_2 = 5$

$\lambda_1 = 2$, $\bar{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\lambda_2 = 5$, $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



7. (10 points) Use LT to solve IVP:

$y' + y = f(t), \quad y(0) = 0, \quad$ where $\quad f(t) = \begin{cases} 0, & \text{for } 0 \le t < 1 \\ 5, & \text{for } t \ge 1 \end{cases}$

$f(t)$: 

$f(t) = 5H(t-1)$

LT: $(s+1)Y = e^{-s} \cdot \frac{5}{s}$, $Y = e^{-s} \frac{5}{s(s+1)}$

$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$Y(s) = 5e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$

$y(t) = 5H(t-1) - 5H(t-1)e^{-(t-1)}$

$\boxed{y(t) = 5H(t-1)\cdot\left(1 - e^{1-t}\right)}$

8. (10 points) Find the general solution of the system

$$y_1' = -5y_1 + y_2$$
$$y_2' = -2y_1 - 2y_2$$

Let $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix}$. Then $\bar{y}' = A\bar{y}$.

$T = -7$, $D = 10 + 2 = 12$, $\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4D}\right]$

$\lambda = \frac{1}{2}[-7 \pm 1]$, $\lambda_1 = -4$, $\lambda_2 = -3$ (e-values).

E-vectors: $(A - \lambda I)\bar{v} = 0$

$\lambda_1 = -4$: $(A + 4I)\bar{v}_1 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 + v_2 \\ -2v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow v_1 = v_2$, $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -3$: $(A + 3I)\bar{v}_2 = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2v_1 + v_2 \\ -2v_1 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow v_2 = 2v_1$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\bar{y}_1 = e^{-4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{y}_2 = e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$W(t) = \det[\bar{y}_1, \bar{y}_2] = \begin{vmatrix} e^{-4t} & e^{-3t} \\ e^{-4t} & 2e^{-3t} \end{vmatrix} = 2e^{-7t} - e^{-7t} = e^{-7t} > 0$

$W(t) \neq 0$ for any $t \in (-\infty, \infty)$ $\Rightarrow$ $\bar{y}_1$ and $\bar{y}_2$ are LI

$\Rightarrow$ $\bar{y}_1$ and $\bar{y}_2$ form the FSS

Hence the gen. sln is $\boxed{\bar{y}(t) = c_1 e^{-4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

9. (15 points) Use the LT to solve the IVP $y'' + 16y = \cos 4t, \quad y(0) = 0, \quad y'(0) = 8$

$$L[y'' + 16y] = L[\cos 4t]$$

$$s^2 Y - 8 + 16Y = \frac{s}{s^2+16}$$

$$Y = \frac{s}{(s^2+16)^2} + \frac{8}{s^2+16}, \quad y(t) = L^{-1}\left[\frac{s}{(s^2+16)^2}\right] + 2L^{-1}\left[\frac{4}{s^2+16}\right]$$

$$L[t\sin 4t] = -F'(s) = -\left[\frac{4}{s^2+16}\right]' = \frac{8s}{(s^2+16)^2}$$

$$\Rightarrow L\left[\tfrac{1}{8} t\sin 4t\right] = \frac{s}{(s^2+16)^2} \Rightarrow$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+16)^2}\right] = \frac{1}{8} t \sin 4t$$

$$L^{-1}\left[\frac{4}{s^2+16}\right] = \sin 4t$$

Hence $\boxed{y(t) = \left(\tfrac{1}{8}t + 2\right)\sin 4t}$