0290 DE Final Exam **Name:** Solutions

(10 problems, 100 points)

functions: t^n, e^{at}, $e^{at}\cos bt$, $t^n a^{at}$, 1, $\sin at$, $e^{at}\sin bt$, $\cos at$

Laplace transforms: $\frac{n!}{(s-a)^{n+1}}$, $\frac{s}{s^2+a^2}$, $\frac{1}{s-a}$, $\frac{s-a}{(s-a)^2+b^2}$, $\frac{n!}{s^{n+1}}$, $\frac{a}{s^2+a^2}$, $\frac{1}{s}$, $\frac{b}{(s-a)^2+b^2}$

Useful formulas: $L[e^{ct}f(t)](s)=F(s-c)$, $L[t^n f(t)](s)=(-1)^n F^{(n)}(s)$, and $L[H(t-c)f(t-c)](s)=e^{-cs}F(s)$.

1. (12 points) Determine whether the given equation is exact. If it is exact, solve it:

$$(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$$

$P(x,y)=\sin y - y\sin x$, $\quad Q(x,y)=\cos x + x\cos y - y$

$\frac{\partial P}{\partial y}=\cos y-\sin x$, $\quad \frac{\partial Q}{\partial x}=-\sin x+\cos y$

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ => The equation is exact.

$F=\int P(x,y)dx = \int(\sin y - y\sin x)dx =$

$=\sin y\int dx - y\int \sin x\,dx = \sin y\cdot x + y\cos x + \varphi(y)$

$F_y = x\cdot\cos y + \cos x + \varphi'(y) = Q = \cos x + x\cos y - y$

$\Rightarrow \varphi'(y)=-y$, $\quad \varphi(y)=-\int y\,dy = -\frac{y^2}{2}$

Solution: $\boxed{x\sin y + y\cos x - \frac{y^2}{2} = C}$

2. (9 points) For the given nonlinear system

$$x' = 8x - y^2$$
$$y' = -6y + 6x^2$$

(a) find both equilibrium points
(b) use the Jacobian to classify each equilibrium point (saddle, spiral sink, etc.)

(a) Equilibrium points: $\begin{cases} 8x - y^2 = 0 \\ -6y + 6x^2 = 0 \end{cases}$ $\quad x = \frac{y^2}{8}$, $\quad -y + x^2 = 0$

$-y + \frac{y^4}{64} = 0$, $\quad y^4 - 64y = 0$, $\quad y(y^3 - 64) = 0$, $\quad y=0, x=0$; $y=4, x=2$

eq. points are: $\boxed{(0,0)}$ and $\boxed{(2,4)}$

(b) Jacobian: $J(x,y) = \begin{bmatrix} 8 & -2y \\ 12x & -6 \end{bmatrix}$

at $(0,0)$: $J(0,0) = \begin{bmatrix} 8 & 0 \\ 0 & -6 \end{bmatrix}$, $\quad D = -48 < 0 \Rightarrow$ $\boxed{\text{saddle point}}$

at $(2,4)$: $J(2,4) = \begin{bmatrix} 8 & -8 \\ 24 & -6 \end{bmatrix}$, $\quad D = -8\cdot 6 + 8\cdot 24 = 8\cdot 18 = 144$

$T = 8 - 6 = 2$, $\quad T^2 - 4D < 0 \Rightarrow$ $\boxed{\text{spiral source}}$.

Both points are generic and their types are preserved for the given nonlinear system

3. (8 points) Find inverse LT of $F(s) = \frac{1-e^{-s}}{s^2}$

$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$, $\quad \boxed{\mathcal{L}^{-1}[F(s)] = t - H(t-1)(t-1)}$. Indeed,

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$,

$\mathcal{L}[H(t-1)(t-1)] = e^{-s}\mathcal{L}(t) = e^{-s}\cdot\frac{1}{s^2} = \frac{e^{-s}}{s^2}$

by the 2nd translation formula.

Hence $\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] = H(t-1)(t-1)$.

4. (10 points) For the given second order differential equation find

(a) the characteristic equation and its roots,
(b) the fundamental set of real-valued solutions (FSS). Prove that these two solutions are linearly independent on the interval of their existence (use Wronskian) and hence they form FSS,
(c) the general real-valued solution

$$x'' - 4x' + 53x = 0$$

(a) $\lambda^2 - 4\lambda + 53 = 0 \qquad \lambda = 2 \pm 7i.$

(b) $y_1 = e^{2t}\cos 7t$, $y_2 = e^{2t}\sin 7t$, $W(t) = \begin{vmatrix} e^{2t}\cos 7t & e^{2t}\sin 7t \\ e^{2t}(2\cos 7t - 7\sin 7t) & e^{2t}(2\sin 7t + 7\cos 7t) \end{vmatrix}$

$= e^{4t}(2\cos 7t \sin 7t + 7\cos^2 7t - 2\cos 7t \sin 7t + 7\sin^2 7t) = 7e^{4t} \neq 0$ for any t.

The interval of existence is $(-\infty, \infty)$; $W(t) \neq 0$ on $(-\infty, \infty)$

$\Rightarrow$ the solutions y_1 & y_2 are LI $\Rightarrow$ they form FSS

(c) $y(t) = c_1 e^{2t}\cos 7t + c_2 e^{2t}\sin 7t$

$$\boxed{y(t) = e^{2t}(c_1\cos 7t + c_2 \sin 7t)}$$

5. (12 points) Find the general solution to the equation

$$y'' + 6y' + 8y = 3e^{-2t} + 2t$$

Char eq: $\lambda^2 + 6\lambda + 8 = 0$, $\lambda_1 = -4$, $\lambda_2 = 2$, $y_h(t) = c_1 e^{-4t} + c_2 e^{-2t}$

Let $3e^{-2t} + 2t = 3f + 2g$, $f = e^{-2t}$, $g = t$.

Particular solutions: $y_f = Ate^{-2t}$, $y_f' = e^{-2t}(A - 2At)$

$y_f'' = e^{-2t}(-2A + 4At - 2A) = e^{-2t}(-4A + 4At)$

$y_f'' + 6y_f' + 8y_f = e^{-2t}(-4A + 4At + 6A - 12At + 8At) = e^{-2t} \cdot 2A = f = e^{-2t}$

$\Rightarrow A = \frac{1}{2}$

$y_g = at + b$, $y_g' = a$, $y_g'' = 0$

$y_g'' + 6y_g' + 8y_g = 6a + 8at + 8b = g = t \Rightarrow a = \frac{1}{8}$, $b = -\frac{3}{32}$

$y_p = 2y_f + 3y_g = te^{-2t} + \frac{3}{8}t - \frac{9}{32}$; gen sln: $\boxed{y(t) = c_1 e^{-4t} + c_2 e^{-2t} + te^{-2t} + \frac{3}{8}t - \frac{9}{32}}$

6. (5 points) Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} &u_t = 4\,u_{xx}, \quad \text{for } t>0,\ 0<x<\pi, \\ &u(0,t) = u(L,t) = 0, \quad \text{for } t>0, \\ &u(x,0) = \sin 2x - \sin 4x, \quad \text{for } 0<x<L \end{aligned}$$

(Hints: (1) This is problem 6 from section 13.2. (2) Show that the steady-state temperature is constant, find this constant. (3) The initial temperature is represented as the Fourier series with two terms and hence the solution will contain two terms only, not infinitely many.)

steady-state $u_s(x)$ satisfies: $\frac{\partial^2 u_s}{\partial x^2} = 0$, $u_s(0) = u_s(L) = 0$; $u_s = ax+b$

$u_s(0) = b$, $u_s(L) = aL+b$

Hence $u_s(x) = 0$. $b=0$, $aL = 0 \Rightarrow a = 0$

$u(x,t) = u_s(x) + v(x,t)$, $v(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$, $L=\pi$

$v(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = g(x) = \sin 2x - \sin 4x - u_s(x) = \sin 2x - \sin 4x$

By the comparison $b_n = 0$ for all n except: $n=2$, $b_2 = 1$

$n=4$, $b_4 = -1$

$n=2$: $e^{-4n^2\pi^2 \cdot t/\pi^2} = e^{-16t}$, $n=4$: $e^{-4n^2\pi^2 t/\pi^2} = e^{-64t}$

Hence $u(x,t) = v(x,t) = \boxed{e^{-16t}\sin 2x - e^{-64t}\sin 4x}$

7. (12 points) Use the LT to solve the IVP $y'' + y = \cos t$, $y(0) = 0$, $y'(0) = 0$

$L[y''+y] = L[\cos t] \iff s^2 Y(s) + Y(s) = \frac{s}{s^2+1}$

$Y(s) = \frac{s}{(s^2+1)^2}$ Apply the derivative formula

$L[t f(t)] = -F'(s)$

$\frac{s}{(s^2+1)^2} = s(s^2+1)^{-2} = -\frac{1}{2}\frac{d}{ds}\left[(s^2+1)^{-1}\right] = \frac{1}{2}\cdot\left[-\left(\frac{1}{s^2+1}\right)'\right]$

$\frac{1}{s^2+1} \leftrightarrow \sin t$ Hence $\boxed{y(t) = \frac{1}{2} t \sin t}$

8. (12 points) Find a FSS for the system

$$y_1' = 2y_1$$
$$y_2' = -6y_1 - 2y_2$$

Let $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 \\ -6 & -2 \end{bmatrix}$. Then $\bar{y}' = A\bar{y}$.

E-values: $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ -6 & -2-\lambda \end{vmatrix} = -(2-\lambda)(\lambda+2) = (\lambda-2)(\lambda+2) = 0$

$\lambda_1 = -2, \ \lambda_2 = 2$

E-vectors: $\lambda_1 = -2$: $(A-\lambda I)\bar{v}_1 = \begin{bmatrix} 4 & 0 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4v_1 \\ -6v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = 0$

$\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$: $(A-\lambda I)\bar{v}_2 = \begin{bmatrix} 0 & 0 \\ -6 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6v_1-4v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -\frac{2}{3}v_2$

take $v_2 = 3$ then $\bar{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$\boxed{\begin{aligned} y_1(t) &= e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ y_2(t) &= e^{2t}\begin{bmatrix} -2 \\ 3 \end{bmatrix} \end{aligned}}$$

$W(t) = \det[y_1(t), y_2(t)] = \begin{vmatrix} 0 & -2e^{2t} \\ e^{-2t} & 3e^{2t} \end{vmatrix} = -2 \neq 0$ for any t

$\Rightarrow y_1$ and y_2 are LI and they form the FSS

9. (10 points) Find the convolution of the two functions $f(t) = t$ and $g(t) = e^t$.

$f * g = \int_0^t u e^{t-u}\,du = e^t \int_0^t u e^{-u}\,du = \left[\begin{matrix}\text{change of} \\ \text{var's: } u = x\end{matrix}\right] =$

$= e^t \int_0^t x e^{-x}\,dx = \left[\begin{matrix}\text{By} & u = x & dv = e^{-x}dx \\ \text{parts:} & du = dx & v = -e^{-x}\end{matrix}\right]$

$= e^t \left[-xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx \right] = e^t \left[-xe^{-x}\Big|_0^t - e^{-x}\Big|_0^t \right]$

$= e^t\left[-te^{-t} - e^{-t} + 1\right] = -t - 1 + e^t = \boxed{e^t - t - 1}$

10. (10 points) For the system

$$y_1' = 0.1y_1 + 2y_2$$
$$y_2' = -2y_1 + 0.1y_2$$

find the type of an equilibrium point and determine is it stable, unstable or asymptotically stable.

The equilibrium point is $(Y_1, Y_2) = (0,0)$.

$A = \begin{bmatrix} 0.1 & 2 \\ -2 & 0.1 \end{bmatrix}$, $D = 0.01 + 4 = 4.01 > 0$, $T = 0.2 > 0$

$T^2 - 4D = 0.04 - 16.04 = -16 < 0$

$\Rightarrow$ it is a $\boxed{\text{spiral source, unstable}}$

Bonus problem: (6 points) Find LT of $f(t) = \sin(3.5t)\cos(3.5t)\, e^{3.5t}$.

$f(t) = \frac{1}{2} \sin 7t \cdot e^{3.5t}$

$L[\frac{1}{2} e^{3.5t} \sin 7t] = \frac{1}{2} \frac{7}{(s-3.5)^2 + 49}$