Math 0413 **Final Exam**

Fall 2015 **Name:** ______________________________

No calculators, no books. Show all your work (no work = no credit). Write neatly.

Part 1. 30 % of the test score.

Give the definition of

(a) (1 point) predicate

(b) (1 point) principle of induction

(c) (1 point) bijective function

(d) (1 point) Archimedean property

(e) (1 point) Cauchy sequence

(f) (1 point) Ratio test for series

Part 2. 70 % of the test score.

1. (7 points) Define a relation $\sim$ on $\mathbb{Z}$ by defining $a \sim b$ to mean $ab \geq 0$.
   Is this an equivalence relation? Support your answer.

2. (a) (3 points) Negate the statement: $\forall \varepsilon > 0\ \exists M \in \mathbb{N}$ such that $\forall n \geq M\ \ |x_n - x| < \varepsilon$.

   (b) (4 points) By using the negation of the previous statement show that $\lim_{n \to \infty} \frac{n-1}{n} \neq 0$

3. (7 points) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}.$

4. (7 points) Let $A = \{2k \in \mathbb{N} : k \geq 5\}$, $B = \{2n - 1 : n \in \mathbb{N}\}$. Show that $|A| = |B|$.

5. (7 points) Let S be an ordered set. Let $B \subset S$ be bounded (above and below). Let $A \subset B$ be a nonempty subset. Suppose that all the inf's and sup's exist. Show that

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

6. (7 points) Is the sequence $\left\{ \frac{n}{3n-1} \right\}$ convergent? Prove your statement using ε, M technique. Find the limit if the sequence is convergent.

7. (7 points) Let $x_n := \dfrac{(-1)^n(n-1)}{2n+1}$. Find $\limsup x_n$ and $\liminf x_n$.

8. (7 points) Prove that $\left\{\dfrac{n+1}{n}\right\}$ is Cauchy using directly the definition of Cauchy sequences.

9. (7 points) Let $\sum x_n$ be a convergent series. Prove that the sequence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = 0$.

10. (7 points) Find if the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+1}}$ is convergent or divergent. Support your answer.