Math 0413 **Sample Final Exam 1**

No calculators, no books. Show all your work (no work = no credit).
Write neatly. Simplify your answers when possible.

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1. Give the definition of
   (a) rule of inference
   (b) power set
   (c) field
   (d) Archimedean property
   (e) reverse triangle inequality
   (f) monotone sequence
   (g) tail of a sequence
   (h) convergence of Cauchy sequence
   (i) absolutely and conditionally convergent series

2. Show that the proposition $P \vee (P \wedge Q) \Leftrightarrow P$ is a tautology.

3. Define a relation $\sim$ on $\mathbb{Z}$ by defining $a \sim b$ to mean $a - b = 5n$ for some integer n. In other words, the difference $a - b$ is divisible by 5.
   Is this an equivalence relation? Support your answer.

4. There is the definition of a continuous function: A function f is continuous if for all x, and for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all y, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
   (a) Write this definition using quantifiers.
   (b) Negate the statement in the part (a), i.e. using quantifiers write a definition of a function that is not continuous.

5. Show that the set of all integer numbers is countable.

6. Let $x, y, z \in F$, where F is an ordered set. Suppose that $x < 0$, $y < z$. Show that $xy > xz$.

7. Prove that if $t > 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n^{-2} < t$.

8. Let A and B be two nonempty bounded sets of real numbers. Let $C := \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that $\sup C = \sup A + \sup B$.

9. Find a number M such that $|x^3 - x^2 + x - 3| \leq M$ for all $-4 \leq x \leq 3$.

10. For $a < b$, construct an explicit bijection from $[a; b]$ to $[-1; 1]$.

11. Let A and B be subsets of $\mathbb{R}$ such that $A \subset B$. Show that $\inf A \geq \inf B$.

12. Is the sequence $\left\{\frac{(-1)^n n}{2n-1}\right\}$ convergent? Prove your statement using ε, M technique.

13. Is the sequence $\left\{\frac{n}{n - \sin(n)}\right\}$ convergent? Prove your statement by applying any method. Find the limit if the sequence is convergent.

14. Suppose that $\{x_n\}$ is a bounded sequence. Let $a_n := \sup\{x_k : k \geq n\}$. Show that a_n is a decreasing sequence.

15. Prove that $\left\{\frac{n-1}{2n}\right\}$ is Cauchy using directly the definition of Cauchy sequences.

16. Find if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$ is convergent or divergent. Support your answer.