Math 0413 **Final Exam**

Summer 2016 **S o l u t i o n s**

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1. By constructing truth table find if the proposition $\sim P \Rightarrow (P \vee \sim Q)$ is a rule of inference or not.

Solution: The proposition is conditional. Its truth table is

P	Q	$\sim P$	$\sim Q$	$P \vee \sim Q$	$\sim P \Rightarrow (P \vee \sim Q)$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

The last column doesn't contain true values only. Hence, the conditional proposition $\sim P \Rightarrow (P \vee \sim Q)$ is not a rule of inference.

2. Show that $|[0,1]| = |\mathbb{R}|$.

Solution: The function $f = \tan(x)$ is a bijection between the interval $(-\pi/2, \pi/2)$ and $\mathbb{R}$.

The function $g = x/\pi + 1/2$ is a bijection between intervals $(-\pi/2, \pi/2)$ and $(0,1)$.

Therefore the function $f \circ g = \tan(x/\pi + 1/2)$ is a bijection between intervals

$(0,1)$ and $\mathbb{R} \;\Rightarrow\; |\,(0,1)\,| = |\mathbb{R}|$.

$(0,1) \subset [0,1] \subset \mathbb{R} \;\Rightarrow\; |\,(0,1)\,| \leq |[0,1]| \leq |\mathbb{R}|$. Therefore $|[0,1]| = |\mathbb{R}|$.

3. Prove that 8 divides the number $3^{2n} - 1$ for any natural n.

Proof: By induction.

Basis statement: For $n = 1$ we have $3^{2n} - 1 = 3^2 - 1 = 8$, 8 divides 8 and the basis statement is true.

Induction step: Assume that the statement is true for n, i.e. $3^{2n} - 1 = 8m$ for some integer m. Then for $n+1$ we have

$3^{2(n+1)} - 1 = 9 \cdot 3^{2n} - 1 = 9 \cdot 3^{2n} - 9 + 8 = 9(3^{2n} - 1) + 8 = 9 \cdot 8m + 8 = 8(9m + 1)$.

Therefore, 8 divides $3^{2(n+1)} - 1$ and the statement is also true for $n+1$.

By the principle of induction, 8 divides $3^{2n} - 1$ for all natural n.

4. Show that $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$.

Solution: See the textbook, corollary 1.2.5, page 31.

5. Show that a convergent sequence has a unique limit.

Solution: See the textbook, proposition 2.1.6, page 49.

6. Using the definition of Cauchy sequence prove or disprove that the sequence $\left\{\dfrac{n^2-2n}{n^2}\right\}$ is Cauchy.

Solution: A sequence is Cauchy if $\forall \varepsilon > 0\ \exists M \in \mathbb{N}$ such that $\forall n, m \geq M$ we have $|x_n - x_m| < \varepsilon$.

For given $\varepsilon > 0$ find (by Archimedean property) $M \in \mathbb{N}$ such that $M\varepsilon > 4$ or $\dfrac{4}{M} < \varepsilon$.

Then for $\forall n \geq M$ and $\forall m \geq M$ we have

$$|x_n - x_m| = \left|\frac{n^2-2n}{n^2} - \frac{m^2-2m}{m^2}\right| = \left|1 - \frac{2}{n} - 1 + \frac{2}{m}\right| = \left|\frac{2}{m} - \frac{2}{n}\right| \leq \frac{2}{m} + \frac{2}{n} \leq \frac{2}{M} + \frac{2}{M} = \frac{4}{M} < \varepsilon.$$

Hence, the sequence is Cauchy.

7. Find if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-n+1}{(-2)^n}$ is conditionally convergent, absolutely convergent, or divergent. Support your answer.

Solution: $n \geq 1 \ \Leftrightarrow\ n-1 \geq 0 \ \Leftrightarrow\ -n+1 \leq 0 \ \Leftrightarrow\ n^3-n+1 \leq n^3 \quad \forall n \in \mathbb{N}$.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n^3-n+1}{(-2)^n}\right| \leq \sum_{n=1}^{\infty} \frac{n^3}{2^n}$

For the right hand side series with $x_n = \dfrac{n^3}{2^n}$ we apply the Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

and the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ is convergent.

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-n+1}{(-2)^n}$ is convergent absolutely by the comparison test.

8. Find if the series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ is conditionally convergent, absolutely convergent, or divergent. Support your answer.

Solution: $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$. It is a p-series with $p = \frac{2}{3} < 1$.

Therefore, the series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ is divergent by the p-test.