Math 0413 **Midterm Exam**

Summer 2016 **S o l u t i o n s**

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1. Show that the proposition $\sim(Q \Rightarrow (P \vee \sim Q))$ is a fallacy.

*Solution:* Truth table:

| $P$ | $Q$ | $\sim Q$ | $P \vee \sim Q$ | $Q \Rightarrow (P \vee \sim Q)$ | $\sim(Q \Rightarrow (P \vee \sim Q))$ |
|---|---|---|---|---|---|
| T | T | F | T | T | F |
| T | F | T | T | T | F |
| F | T | F | F | F | T |
| F | F | T | T | T | F |

The last column doesn't contain true values only. Hence, the proposition $\sim(Q \Rightarrow (P \vee \sim Q))$ is a fallacy.

2. Negate the statement $\forall x \in \mathbb{R}\ \exists a, b \in \mathbb{R}$ such that if $x < a$ then $x < b$.

(Note: neither the statement nor its negation has to be true).

*Solution:* The negation is

$$\sim (\forall x \in \mathbb{R}\ \exists a, b \in \mathbb{R} :\ x < a\ \Rightarrow\ x < b)$$

$$(\exists x \in \mathbb{R} :)\ \sim (\exists a, b \in \mathbb{R}\ x < a\ \Rightarrow\ x < b)$$

$$(\exists x \in \mathbb{R} :)\ (\forall a, b \in \mathbb{R})\ \sim (x < a\ \Rightarrow\ x < b)$$

$$(\exists x \in \mathbb{R} :)\ (\forall a, b \in \mathbb{R})\ (x < a)\ \wedge \sim (x < b)$$

$$(\exists x \in \mathbb{R} :)\ (\forall a, b \in \mathbb{R})\ (x < a)\ \wedge\ (x \geq b)$$

Finally,

$$\exists x \in \mathbb{R} \text{ such that } \forall a, b \in \mathbb{R}\ \ b \leq x < a$$

3. For $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ define $(a, b) \sim (c, d)$ to mean that $a + d = b + c$.

Is this an equivalence relation? Support your answer.

*Solution:* The definition of the given relation should be read as

$(a, b) \sim (c, d) \quad \Leftrightarrow \quad a + d = b + c$

Reflexivity: $a + b = b + a \quad \Rightarrow \quad (a, b) \sim (a, b)$.

Symmetry:
$(a, b) \sim (c, d) \quad \Leftrightarrow \quad a + d = b + c \quad \Leftrightarrow \quad b + c = a + d \quad \Leftrightarrow \quad c + b = d + a \quad \Leftrightarrow \quad (c, d) \sim (a, b)$.

Transitivity: $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f) \quad \Leftrightarrow \quad a + d = b + c$ and $c + f = d + e$.
If we add $e$ to the first equation and $a$ to the second equation we obtain
$a + d + e = b + c + e$ and $a + c + f = a + d + e$
which gives $a + c + f = b + c + e$ or $a + f = b + e \quad \Leftrightarrow \quad (a, b) \sim (e, f)$.

The relation is reflexive, symmetric, and transitive. Hence it is an equivalence relation.

4. Let $A$ be a set and $\mathscr{P}(A)$ be its power set. Show that a function $f : A \to \mathscr{P}(A)$ is not a surjection.

*Solution:* See the textbook, proposition 0.3.27, page 21.

5. Let $F$ be an ordered field and $x, y, z \in F$. Show that $x \neq 0 \quad \Rightarrow \quad x^2 > 0$.

In a proof you may use the following properties of $F$:
(1) $x > 0 \quad \Leftrightarrow \quad -x < 0$; (2) $x > 0, y < z \quad \Rightarrow \quad xy < xz$; and (3) $x < 0, y < z \quad \Rightarrow \quad xy > xz$.

*Proof:* See the textbook, proposition 1.1.8, part (iv), page 27.

6. By using the Archimedean property of real numbers prove that

if $t \in \mathbb{R},\ t < 0$ then $\exists n \in \mathbb{N}$ such that $-\dfrac{1}{n} > t$.

*Proof:* By property (i) of the proposition 1.1.8 we have $t < 0 \quad \Leftrightarrow \quad -t > 0$
Consider the Archimedean property with $x = -t > 0$, $y = 1$. Then $\exists n \in \mathbb{R}$ such that $n(-t) > 1$.
We multiply both sides of the last inequality by a positive $\dfrac{1}{n}$ to get $-t > \dfrac{1}{n}$ or $\dfrac{1}{n} < -t$
Multiplication by (-1) and the property (iii) of the proposition 1.1.8 give $-\dfrac{1}{n} > t$.