Math 0413 **Midterm Exam**

Spring 2018 **S o l u t i o n s**

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1. Give the definition of least upper bound property of a set.

Solution: An ordered set A has the least upper bound property if every non-empty subset $E \subset A$ that is bounded above has a least upper bound in A.

2. Negate the statement "For every $a, b \in \mathbb{R}$ with $a < b$ there is an $r \in \mathbb{Q}$ with $a < r < b$".

Solution: "There are $a, b \in \mathbb{R}$ with $a < b$ such that for all $r \in \mathbb{Q}$ we have $a \geq r$ or $r \geq b$".

3. Suppose $\sim$ is an equivalence relation on a set A. Show that $\forall a, b \in A \quad [a] \cap [b] \neq \varnothing$ implies $[a] = [b]$, where $[a]$ denotes the equivalence class of the element a.

Solution: $[a] \cap [b] \neq \varnothing \quad \Rightarrow \quad \exists y \in [a] \cap [b]$, that is, $a \sim y$ and $b \sim y \quad \Rightarrow \quad a \sim b$ by transitivity and $b \sim a$ by reflexivity.

We need to show that the two sets $[a]$ and $[b]$ are equal.

If $x \in [a]$, then $x \sim a$, $a \sim b \quad \Rightarrow \quad x \sim b$ by transitivity, that is, $x \in [b]$.

Conversely, if $x \in [b]$, then $x \sim b$, $b \sim a \quad \Rightarrow \quad x \sim a$ by transitivity, that is, $x \in [a]$.

Therefore $[a] = [b]$.

4. Let $A = \left\{ \frac{1}{n} : \ n \in \mathbb{N} \right\}$, $B = \{2k : \ k \in \mathbb{N}\}$. Show that $|A| = |B|$.

Solution: Let $f : \ A \rightarrow B$ is defined by $f(x) = \dfrac{2}{x}$. Then f is a bijection.

f is a surjection: Let $n \in B$. Then $\exists k \in \mathbb{N}$ such that $n = 2k$. Hence $n \in \mathbb{N}$. Take $x = \dfrac{2}{n}$.

Obviously $x \in A$ since $n \geq 2$. $f(x) = \dfrac{2}{x} = \dfrac{2}{2/n} = n \in B$.

So, for any element n in B there is an element x in A such that $f(x) = n$. Therefore f is a surjection.

f is an injection: Let $x_1, x_2 \in A$. Then

$$f(x_1) = f(x_2) \quad \Rightarrow \quad \frac{2}{x_1} = \frac{2}{x_2} \quad \Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{2} \quad \Rightarrow \quad x_1 = x_2.$$

Therefore f is an injection.

So f is a bijection and it follows that $|A| = |B|$.

5. Consider the increasing sequence of real numbers $x_1 = 1$ and $x_{n+1} = \sqrt{1 + 2x_n}$ for $n \geq 1$.
Use the Principle of Mathematical Induction to show that $x_n < 4 \ \forall n \geq 1$.

Proof: By induction. Define the statement $P(n)$ as $x_n < 4$.

Basis statement $P(1)$: $x_1 = 1 < 4$ and the basis statement is true.

Induction step: Assume that the statement $P(n)$ is true, i.e. $x_n < 4$.
Then for $n+1$ we have $x_{n+1} = \sqrt{1 + 2x_n} < \sqrt{1 + 2 \cdot 4} = \sqrt{9} = 3 < 4$.
Therefore, $P(n+1)$ is true.
By the principle of induction, $P(n)$ is true for all natural n, i.e. $x_n < 4 \ \forall n \geq 1$.

6. Prove that if $A = \left\{1 - \frac{1}{n},\ n \in \mathbb{N}\right\}$ then $\sup A = 1$.

Proof: $0 \in A \Rightarrow A \neq \varnothing$.
$1 - \frac{1}{n} < 1 \Rightarrow$ 1 is an upper bound of A.
$A \in \mathbb{R}$ and the set $\mathbb{R}$ has the least upper bound property $\Rightarrow b = \sup A$ exists in $\mathbb{R}$.
1 is an upper bound of $A \Rightarrow b \leq 1$.
Assume $b \neq 1$. Then $b < 1 \Rightarrow 1 - b > 0$.
Consider the Archimedean property with $x = 1 - b > 0$ and $y = 1$.
Then $\exists n \in \mathbb{N}$ such that $n(1-b) > 1 \Rightarrow 1 - b > \dfrac{1}{n} \Rightarrow b < 1 - \dfrac{1}{n} \in A$
$\Rightarrow$ b is not an upper bound of A. A contradiction! $\Rightarrow$ the assumption $b \neq 1$ was wrong
$\Rightarrow b = 1 \Leftrightarrow \sup A = 1$.

Alternative proof: Define $B = \left\{\frac{1}{n},\ n \in \mathbb{N}\right\}$. Then $A = 1 + (-1)B$.

By corollary 1.2.5 $\inf B = 0$.

By proposition 1.2.6

$\sup A = \sup\left(1 + (-1)B\right) = 1 + \sup\left((-1)B\right) = 1 + (-1) \cdot \inf B = 1 + (-1) \cdot 0 = 1.$