Math 0413 **Final Exam**

Spring 2018 **Solutions**

---

1. By constructing truth table find if the proposition $\sim P \Rightarrow (P \vee \sim Q)$ is a rule of inference or not.

Solution: The proposition is conditional. Its truth table is

P	Q	$\sim P$	$\sim Q$	$P \vee \sim Q$	$\sim P \Rightarrow (P \vee \sim Q)$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

The last column doesn't contain true values only. Hence, the conditional proposition $\sim P \Rightarrow (P \vee \sim Q)$ is not a rule of inference.

2. Let $f\colon A \to B$ be a function and $C \subset B$. Show that

$$f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$$

Solution: Let $x \in f^{-1}(B \setminus C) \;\Rightarrow\; f(x) \in B \setminus C \;\Rightarrow\; f(x) \notin C$

$\Rightarrow\; x \notin f^{-1}(C) \;\Rightarrow\; x \in A \setminus f^{-1}(C) \;\Rightarrow\; f^{-1}(B \setminus C) \subset A \setminus f^{-1}(C)$

Now let $x \in A \setminus f^{-1}(C) \;\Rightarrow\; x \notin f^{-1}(C) \;\Rightarrow\; f(x) \notin C \;\Rightarrow\; f(x) \in B \setminus C$

$\Rightarrow\; x \in f^{-1}(B \setminus C) \;\Rightarrow\; A \setminus f^{-1}(C) \subset f^{-1}(B \setminus C)$.

Therefore, $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$.

3. Show that $|[0,1]| = |\mathbb{R}|$.

Solution: The function $f = \tan(x)$ is a bijection between the interval $(-\pi/2, \pi/2)$ and $\mathbb{R}$.
The function $g = \pi x - \frac{\pi}{2}$ is a bijection between intervals $(0,1)$ and $(-\pi/2, \pi/2)$.
Therefore the function $f \circ g = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection between intervals $(0,1)$ and $\mathbb{R} \;\Rightarrow\; |\,(0,1)\,| = |\mathbb{R}|$.
$(0,1) \subset [0,1] \subset \mathbb{R} \;\Rightarrow\; |\,(0,1)\,| \le |[0,1]| \le |\mathbb{R}|$. Therefore $|[0,1]| = |\mathbb{R}|$.

4. (a) Give the definition of the principle of induction.

   Solution: See the textbook, page 16.

   (b) Prove that 8 divides the number $3^{2n} - 1$ for any natural n.

   Proof: By induction.

   Basis statement: For $n = 1$ we have $3^{2n} - 1 = 3^2 - 1 = 8$, 8 divides 8 and the basis statement is true.

   Induction step: Assume that the statement is true for n, i.e. $3^{2n} - 1 = 8m$ for some integer m. Then for $n+1$ we have

   $3^{2(n+1)} - 1 = 9 \cdot 3^{2n} - 1 = 9 \cdot 3^{2n} - 9 + 8 = 9(3^{2n} - 1) + 8 = 9 \cdot 8m + 8 = 8(9m + 1).$

   Therefore, 8 divides $3^{2(n+1)} - 1$ and the statement is also true for $n + 1$.

   By the principle of induction, 8 divides $3^{2n} - 1$ for all natural n.

5. Let $a, b, x \in \mathbb{R}$. Prove that if $x \geq a$ $\forall a < b$ then $x \geq b$.

Solution: Proof by contradiction. Assume that $x \geq b$ is not true. Then $x < b$ is true. $\mathbb{Q}$ is dense in $\mathbb{R}$ $\Rightarrow$ $\exists a \in \mathbb{Q}$ such that $x < a < b$. This contradicts to the statement that $x \geq a$ $\forall a < b$. Therefore the assumption $x \geq b$ was wrong. Then $x < b$ is true.

An alternative solution: Proof by contradiction. Assume that $x \geq b$ is not true.

Then $x < b$ is true $\Rightarrow$ $x + b < b + b$ $\Rightarrow$ $x + b < 2b$ $\Rightarrow$ $\dfrac{x+b}{2} < b.$

Take $a = \dfrac{x+b}{2}$ $\Rightarrow$ $a < b.$

We also have $x < b$ $\Rightarrow$ $x + x < x + b$ $\Rightarrow$ $2x < x + b$ $\Rightarrow$ $x < \dfrac{x+b}{2}$ $\Rightarrow$ $x < a.$

That contradicts to the fact that if $a < b$ then $x \geq a$. Hence the assumption $x < b$ was wrong.

Therefore $x \geq b$ must be true.

6. Show that a convergent sequence has a unique limit.

Solution: See the textbook, proposition 2.1.6, page 49.

7. (a) Give the definition of a convergent sequence.

Solution: A sequence is convergent to its limit x if $\forall \varepsilon > 0\ \exists M \in \mathbb{N}$ such that $\forall n \geq M$ we have $|x_n - x| < \varepsilon$.

(b) Is the sequence $\left\{ \dfrac{n}{5n+1} \right\}$ convergent? Support your answer by using ε and M from the definition of a convergent sequence. If the sequence is convergent then find the limit.

Solution: For given $\varepsilon > 0$ find (by Archimedean property) $M \in \mathbb{N}$ such that $M\varepsilon > \frac{1}{25}$ or $\dfrac{1}{25M} < \varepsilon$.

Then for $\forall n \geq M$ we have

$$|x_n - \tfrac{1}{5}| = \left| \frac{n}{5n+1} - \frac{1}{5} \right| = \left| \frac{5n - 5n - 1}{5(5n+1)} \right| = \left| \frac{-1}{5(5n+1)} \right| = \tfrac{1}{5} \cdot \frac{1}{5n+1} \leq \tfrac{1}{5} \cdot \frac{1}{5n} = \tfrac{1}{25} \cdot \frac{1}{n}$$

$$\leq \tfrac{1}{25} \cdot \frac{1}{M} < \varepsilon.$$

Hence, the sequence is convergent. Its limit is $\frac{1}{5}$.

8. (a) Give the definition of a Cauchy sequence.

Solution: A sequence $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0\ \exists M \in \mathbb{N}$ such that $\forall n, m \geq M$ we have $|x_n - x_m| < \varepsilon$.

(b) Using the definition of Cauchy sequence prove or disprove that the sequence $\left\{ \dfrac{n^2 - 7n}{n^2} \right\}$ is Cauchy.

Solution: For given $\varepsilon > 0$ find (by Archimedean property) $M \in \mathbb{N}$ such that $M\varepsilon > 14$ or $\dfrac{14}{M} < \varepsilon$.

Then for $\forall n \geq M$ and $\forall m \geq M$ we have

$$|x_n - x_m| = \left| \frac{n^2 - 7n}{n^2} - \frac{m^2 - 7m}{m^2} \right| = \left| 1 - \frac{7}{n} - 1 + \frac{7}{m} \right| = \left| \frac{7}{m} - \frac{7}{n} \right| \leq \frac{7}{m} + \frac{7}{n} \leq \frac{7}{M} + \frac{7}{M} = \frac{14}{M} < \varepsilon.$$

Hence, the sequence is Cauchy.

9. Show that if a sequence $\{x_n\}$ is convergent then it is Cauchy.

Solution: See the textbook, page 74.

10. Find if the series $\sum_{n=1}^{\infty} \frac{n^4 - 3n + 1}{(-2)^n}$ is conditionally convergent, absolutely convergent, or divergent. Support your answer.

Solution: $n \geq 1 \;\Rightarrow\; 3n \geq 1 \;\Leftrightarrow\; 3n - 1 \geq 0 \;\Leftrightarrow\; -3n + 1 \leq 0$

$\Leftrightarrow\; n^4 - 3n + 1 \leq n^4 \quad \forall n \in \mathbb{N}.$

We also have $n^4 - 3n + 1 > 0 \quad n \geq 2.$

Therefore, $\sum_{n=2}^{\infty} \left| \frac{n^4 - 3n + 1}{(-2)^n} \right| \leq \sum_{n=2}^{\infty} \frac{n^4}{2^n}$

For the right hand side series with $x_n = \frac{n^4}{2^n}$ we apply the Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n\to\infty} \frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{n^4} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^4 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

and the series $\sum_{n=2}^{\infty} \frac{n^3}{2^n}$ is convergent.

Therefore, the series $\sum_{n=2}^{\infty} \left| \frac{n^4 - 3n + 1}{(-2)^n} \right|$ is convergent by the comparison test

$\Rightarrow \sum_{n=2}^{\infty} \frac{n^4 - 3n + 1}{(-2)^n}$ is convergent absolutely

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^4 - 3n + 1}{(-2)^n}$ is convergent absolutely.

An alternative solution: Ratio test: $L = \lim_{n\to\infty} \left| \frac{(n+1)^4 - 3(n+1) + 1}{(-2)^{n+1}} \cdot \frac{(-2)^n}{n^4 - 3n + 1} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^4 - 3(n+1) + 1}{n^4 - 3n + 1} \cdot \frac{(-2)^n}{(-2)^{n+1}} \right| = \frac{1}{2} < 1.$$

$\Rightarrow$ the series $\sum_{n=1}^{\infty} \frac{n^4 - 3n + 1}{(-2)^n}$ is convergent absolutely.