Final Exam

Summer 2011 Math 1180

100 points total **Your name:** ______________________________

No calculators. Show all your work (no work = no credit) and explain every step. Write neatly.

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1. [10 points] Solve the following linear system by using Cramer's Rule:

$$\begin{array}{rcrcrcl} 4x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 3 \\ -2x_1 & - & x_2 & + & 2x_3 & = & 4 \end{array}$$

2. [10 points] Let $\bar{\mathsf{u}} = [1, 0, 5, -2]$ and $\bar{\mathsf{v}} = [3, 5, 7, 1]$. Find $\mathrm{proj}_{\bar{\mathsf{v}}}\bar{\mathsf{u}}$, $\mathrm{perp}_{\bar{\mathsf{v}}}\bar{\mathsf{u}}$ and the cosine of the angle between these vectors. Simplify obtained numbers.

3. [10 points] Write the vector $\bar{\mathsf{u}} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$ as a linear combination of vectors $\bar{\mathsf{v}}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\bar{\mathsf{v}}_2 = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$, and $\bar{\mathsf{v}}_3 = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$.

4. [10 points] Find the eigenpairs and orthogonal bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

5. [10 points] Use the Gram-Schmidt process to transform the vectors $\bar{\mathsf{x}}_1 = [1, 0, 1]$, $\bar{\mathsf{x}}_2 = [1, 2, -2]$, and $\bar{\mathsf{x}}_3 = [2, -1, 1]$ into an orthonormal basis for $\mathbb{R}^3$.

6. [10 points] Describe geometrically the space $V = \text{span}(\bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2)$, where $\bar{\mathsf{x}}_1 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$, $\bar{\mathsf{x}}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ are vectors in $\mathbb{R}^3$ and find a vector in $V^{\perp}$.

7. (a) [10 points] Let F be the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$F([x, y, z]) = [x, y].$$

Is F linear? Prove or disprove this.

(b) [10 points] Let T be the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T([x, y]) = [2x + 1, y - 2].$$

Is T linear? Prove or disprove this.

8. [10 points] Find a diagonal matrix D and an orthogonal matrix Q such that $Q^T AQ = D$ if

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Calculate QDQ^T.

9. [10 points] Show that $p_1(x) = 2x^2 + x$, $p_2(x) = x^2 + 2x + 1$, and $p_3(x) = x + 2$ form a linearly independent set in $\mathscr{P}_2$ and hence its basis. Find the coordinate vector of $p(x) = x^2 + 1$ with respect to this basis.

bonus problem [10 points extra] Let T be the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T([x, y]) = [2x + y, -x + 3y].$$

Let T^{-1} be the inverse transformation of T given by

$$T^{-1}([x, y]) = [ax + by, cx + dy].$$

Find a, b, c, and d.