Quiz 2

Summer 2011

Math 1180 **Your name:** Solutions

1. [5 points] Use the Gauss-Jordan method to find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Sln

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3R_2 - R_1} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & -1 & 3 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{9R_3 - R_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & -1 & 3 & 0 \\ 0 & 0 & 27 & 1 & -3 & 9 \end{array}\right] \xrightarrow{\frac{1}{3}R_1,\ \frac{1}{9}R_2,\ \frac{1}{27}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{9} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{27} & -\frac{1}{9} & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{9} & \frac{1}{3} & 0 \\ \frac{1}{27} & -\frac{1}{9} & \frac{1}{3} \end{bmatrix}$$

2. [5 points] Compute B^8 and B^9 if

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Sln:

$$B^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}-\frac{1}{2} & -\frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^4 = (B^2)^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$$

$$B^8 = (-I_2)^2 = I_2$$

$$B^9 = B^8 B = I B = B$$

$$\boxed{B^8 = I_2 \quad , \quad B^9 = B}$$

3. [5 points] We know that $(AB)^T = B^T A^T$. Using induction prove, that for all $n \geq 1$,

$$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T.$$

Sln: It is true for $n=1$ (obvious) and for $n=2$: $(A_1 A_2)^T = A_2^T A_1^T$ (given) (basis step)

Assume that the statement is true for $n \geqslant 2$ (induction hypothesis)

Then for $n+1$ (induction step)

$$(A_1 A_2 \cdots A_n A_{n+1})^T = A_{n+1}^T (A_1 A_2 \cdots A_n)^T =$$

$$= A_{n+1}^T A_n^T \cdots A_2^T A_1^T \quad \Rightarrow \text{ true for } n+1$$

By the induction the statement is true for any $n \geqslant 1$.

bonus problem [5 points extra] If A is any matrix, to what is $\text{tr}(AA^T)$ equal?

Let A be $n \times m$ matrix and $\bar{A}_i$ be its $1 \times m$ i^{th} row vector. Then

$$AA^T = \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \vdots \\ \bar{A}_n \end{bmatrix} \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_n \end{bmatrix} = \begin{bmatrix} \|\bar{A}_1\|^2 & & * \\ & \|\bar{A}_2\|^2 & \\ * & & \ddots \\ & & & \|A_n\|^2 \end{bmatrix}$$

$$\Rightarrow \text{tr}(AA^T) = \|\bar{A}_1\|^2 + \|\bar{A}_2\|^2 + \cdots + \|\bar{A}_n\|^2 = \sum_{i=1}^{n} \|\bar{A}_i\|^2$$

but $\bar{A}_i = [a_{i1}, a_{i2}, \cdots, a_{im}]$ and

$$\|A_i\|^2 = \sum_{j=1}^{m} a_{ij}^2$$

Hence $\text{tr}(AA^T) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2$

It is the sum of squares of all nm entries of A