Quiz 4

Summer 2011

Math 1180 **Your name:** Solutions

1. [5 points] Show that the vectors $\bar{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\bar{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^3$. Then find the coordinate vector $[\bar{w}]_{\mathcal{B}}$ of $\bar{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

Sln: See example 5.4, page 369

2. [5 points] Determine whether

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 + 2 =$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$$

e-values are distinct. By theorem 4.25 A is diagonalizible.

<u>e-vectors:</u>

$\lambda_1 = 2$: $(A - \lambda_1 I)\bar{x}_1 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0 \Leftrightarrow x_{11} = 2x_{12}$

$\Rightarrow \bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$: $(A - \lambda_2 I)\bar{x}_2 = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0 \Leftrightarrow x_{21} = x_{22}$

$\Rightarrow \bar{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\det P = 2 - 1 = 1 \neq 1$ $\Rightarrow$

$\Rightarrow$ P is invertible. $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = D$.

See example 5.10, page 378

3. [5 points] Let W be the subspace spanned by the vectors $\bar{\mathbf{w}}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $\bar{\mathbf{w}}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$. Find a basis for $W^{\perp}$.

W is a subspace of $\mathbb{R}^3$. Consider

$A = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$ made of $\bar{w}_1$ and $\bar{w}_2$

$W^{\perp} = (col(A))^{\perp} = null(A^T)$

$$[A^T | 0] = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right] \xrightarrow{2R_1 - R_2} \left[\begin{array}{ccc|c} 0 & 2 & -5 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right]$$

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $W^{\perp}$ if $\begin{array}{l} 2x_2 - 5x_3 = 0 \\ 4x_1 + x_3 = 0 \end{array}$

$x_2 = \frac{5}{2} x_3$, $x_1 = -\frac{1}{4} x_3$. Set $x_3 = 4$. Then

$\bar{x} = \begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix}$, $W^{\perp} = span(\bar{x})$

$\Rightarrow$ $\bar{x}$ is a basis for $W^{\perp}$. Answer: $\begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix}$

bonus problem [5 points extra] Show that the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar.

$A \sim B$ if there is an invertible P s.t.

$AP = PB$.

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$AP = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$, $PB = \begin{bmatrix} a+b & b \\ c+d & d \end{bmatrix}$

$\Rightarrow b = c, \ d = 0$.

Hence $P = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$. $\det P = -b^2 \neq 0$ if $b \neq 0 \Rightarrow P$ is invertible.

P exists $\Rightarrow A \sim B$.